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Dynamics in dumbbell domains I. Continuity of the set of equilibria

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Abstract

We analyze the dynamics of a reaction–diffusion equation with homogeneous Neumann boundary conditions in a dumbbell domain. We provide an appropriate functional setting to treat this problem and, as a first step, we show in this paper the continuity of the set of equilibria and of its linear unstable manifolds.

MSC: primary 35B40; secondary 35K55, 35B25

Keywords: Reaction–diffusion equation; Stationary solutions; Dumbbell domains; Continuity

1. Introduction

This paper is the first one of a series of articles whose final objective is to address the problem of the behavior of the asymptotic nonlinear dynamics of a reaction–diffusion equation when the domain where the equation is posed undergoes a singular perturbation.

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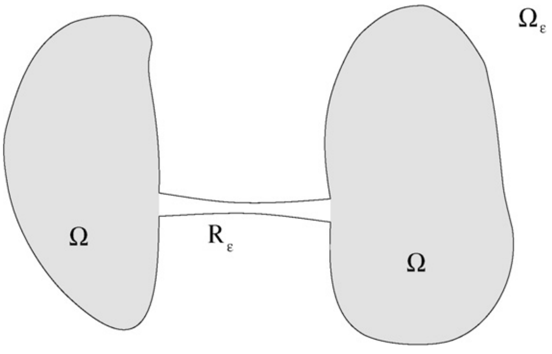

Fig. 1. Dumbbell domain.

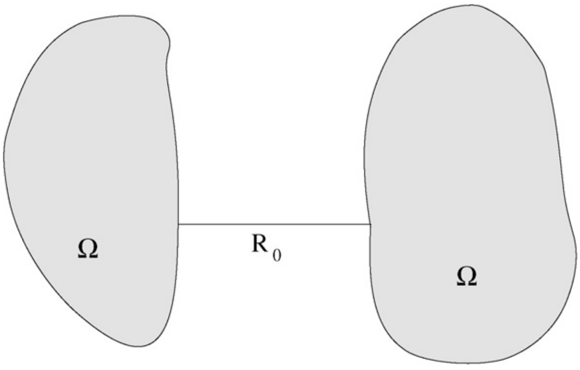

Fig. 2. Limit “domain.”

In particular, we consider the evolution equation of parabolic type of the form

$$\begin{cases} u_t - \Delta u + u = f(u), & x \in \Omega_\epsilon,\ t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded smooth domain, $\epsilon \in (0, 1]$ is a parameter, $\frac{\partial}{\partial n}$ is the outside normal derivative and $f : \mathbb{R} \to \mathbb{R}$ is a dissipative nonlinearity.

The domain Ω_ϵ is a dumbbell-type domain consisting of two disconnected domains, that we will denote by Ω, joined by a thin channel, R_ϵ, which degenerates to a line segment as the parameter ϵ approaches zero, see Fig. 1.

Under standard dissipative assumption on the nonlinearity f of the type

$$\limsup_{|s| \to +\infty} f(s)/s < 1,$$

for fixed $\epsilon \in (0, 1]$, Eq. (1.1) has an attractor $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$.

On the other hand, passing to the limit as $\epsilon \to 0$, the limit “domain” will consist of the domain Ω_0 and a line in between, see Fig. 2.

And the limit equation is

$$\begin{cases} w_t - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega, \\ v_t - Lv + v = f(v), \quad s \in R_0, \\ v(p_0) = w(p_0), \qquad v(p_1) = w(p_1), \end{cases} \tag{1.2}$$

where w is a function that lives in Ω and v lives in the line segment R_0. Moreover, L is a differential operator which depends on the geometry of the channel R_ϵ, more exactly, on the way the channel R_ϵ collapses to the segment line R_0. For instance, in two dimensions, if the channel $R_\epsilon = \{(x, y)\colon\ 0 < x < 1,\ 0 < y < \epsilon g(x)\}$, then $Lv = \frac{1}{g}(gv_x)_x$. For other channels, the operator L needs to be calculated explicitly. We also denote by p_0 and p_1 the points where the line segment touches the domain Ω. Again, this system has an attractor $\mathcal{A}_0$ in $H^1(\Omega) \times H^1(R_0)$.

We are interested in understanding the relation between the attractors $\mathcal{A}_\epsilon$, $\epsilon \in (0, 1]$, and $\mathcal{A}_0$. With the results of this paper and with [7], we will show that this family of attractors is upper semicontinuous at $\epsilon = 0$ in certain topology, and if all the equilibria in $\mathcal{A}_0$ are hyperbolic, then the attractors are continuous, that is, upper and lower semicontinuous.

In appropriate functional spaces, we will see that problem (1.1) can be written as an evolutionary equation of the type

$$\begin{cases} u_t + A_\epsilon u = F_\epsilon(u), \quad t > 0, \\ u(0) \in X_\epsilon \end{cases} \tag{1.3}$$

for certain family of spaces X_ϵ. Also, problem (1.2) can also be written as

$$\begin{cases} u_t + A_0 u = F_0(u), \quad t > 0, \\ u(0) \in X_0 \end{cases} \tag{1.4}$$

in a certain space X_0.

In this paper, we will work out an appropriate functional setting to treat a broad class of perturbation problems which, in particular, will include the case of the singular perturbation problem of the dumbbell domain, that is, problems (1.3) and (1.4). This functional setting will make use of several concepts like the concept of convergence for a sequence $\{u_\epsilon\}_{\epsilon\in(0,1]}$ where u_ϵ belongs to different spaces X_ϵ for each $\epsilon \in (0, 1]$, an appropriate concept of compactness for families living in different spaces and the concept of "compact convergence" as the key concept to treat the behavior of compact operators in different spaces. This setting is developed mainly in Sections 4 and 5.

The program that we will follow to prove the continuity of the attractors is divided in two parts. The first one, which is addressed in this paper, consists in proving the continuity of the equilibria and, in case the equilibrium is hyperbolic, obtaining the continuity of its linear unstable manifolds. Hence:

(1) We will first show the convergence of the resolvent operators, that is will show that A_ϵ^{-1} converge in an appropriate way to A_0^{-1}, see Proposition 2.7. This is a key point to all the analysis.
(2) Writing the stationary problem as a fixed point problem, that is, u_ϵ is an equilibrium solution of (1.3) (respectively u_0 is an equilibrium of (1.4)) if $u_\epsilon = A_\epsilon^{-1} F_\epsilon(u_\epsilon)$ (respectively $u_0 =$

$A_0^{-1}F_0(u_0)$) and with the convergence of the linear resolvent operators, we will show the convergence of the equilibria. Moreover, we will show that if an equilibrium of the limit problem is hyperbolic, then it is isolated and there exists one and only one equilibrium of the perturbed problem nearby, see Theorem 2.3.

(3) With the convergence of the resolvent operators and with the convergence of the equilibria, we will also show the convergence of the resolvent operators of the linearizations around the equilibria, that is the convergence of $(A_\epsilon - F_\epsilon'(u_\epsilon) + \lambda)^{-1}$ to $(A_0 - F_\epsilon'(u_0) + \lambda)^{-1}$, for some λ large enough. For the case where the equilibrium is hyperbolic, this will imply the convergence of the linear unstable manifolds, see Theorem 2.5.

The second part, which is developed in [7] consists in proving the convergence of the linear and nonlinear semigroups and the nonlinear unstable manifolds of the equilibria:

(4) With the convergence of the resolvent operators A_ϵ^{-1} to A_0^{-1} we will show, with a Trotter–Kato-type formula, the convergence of the linear semigroups $e^{-A_\epsilon t}$ to $e^{-A_0 t}$.
(5) With the variation of constants formula we will show the convergence of the nonlinear semigroups. Once this is accomplished, the upper semicontinuity of attractors is easily obtained.
(6) Assuming the equilibria are all hyperbolic, with the convergence of the linear unstable manifolds and with a similar argument as it is done in [6] we will be able to show the convergence of the local nonlinear unstable manifolds. Using now that the system is gradient we will easily show that the attractors are lower semicontinuous and therefore continuous.

This agenda, or variants of it, has been proved to be successful when addressing the behavior of the long time dynamics in different perturbation problems. It is based in a careful study of the behavior of the linear parts under the perturbation considered and this information is translated into the nonlinear dynamics through the variation of constants formula. In [5], a general approach to obtain upper semicontinuity of attractors following this approach is explained. Also, a similar technique to get the upper semicontinuity was used in [38] for the case of thin domains with "holes." In [1,6] this same technique is used to obtain the continuity (upper and lower semicontinuity) of the attractors of reaction–diffusion equations with Dirichlet and Neumann boundary conditions when the domain is perturbed. Actually, in [6], the only condition imposed in the perturbed domains is the spectral convergence of the linear operators. Inspired by the works [1,6] a general scheme to treat the continuity of the attractors of semilinear parabolic problems is developed in [9]. We also refer to [15,17] for general theorems guaranteeing the lower semicontinuity of the attractors.

The "dumbbell domain" problem has been considered before by many authors. It appears naturally as the counterpart of a convex domain in the following situation. It is well known that in a convex domain the stable stationary solutions to (1.1) are necessarily spatially constant, see [10,32]. This is due to the fact that gradients of temperature can be diffused in the shortest path (the line segment between the two points with different temperatures). One way to produce "patterns," that is, stable stationary solutions which are not spatially constant, is to consider domains which makes it difficult for the heat to flow from one part of the domain to the other, making a constriction in the domain. It becomes natural to consider dumbbell like domains as a prototype domain to produce this "patterns." With this purpose the so-called dumbbell domains with a bistable nonlinearity of the type $f(u) = u - u^3$ was considered in [35].

It seems clear that when passing from a convex domain to a nonconvex domain (like a dumbbell domain) some kind of bifurcation of equilibria appears. This aspect was studied in [19,40].

In several works at the end of the 80's [22–26] and beginning of the 90's [27,28] Jimbo made a detail analysis of the behavior of linear and semilinear elliptic problems in dumbbell-type domains with two important characteristics: (1) the dimension is larger or equal to three and (2) the channel R_ϵ is a straight cylindrical channel. His analysis is based on a very careful and detailed study of the L^∞ norm of the eigenfunctions of the Laplace operator with Neumann boundary conditions in the junction of the channel with the fixed part of the domain.

With regards to the spectral behavior of the Laplace operator in dumbbell domains we refer to [25] for a straight cylindrical channel and to [2–4] for more general channels. See also [16] for a survey on results on the behavior of eigenvalues under perturbations of the domain and [21] for a general method to treat regular perturbations of the domain. Recently there has been a study of the rates of the eigenvalues of the dumbbell domains in dimension 3 with a cylindrical channel in [14]. Also, in [11] the authors analyze spectral properties in a multidimensional structure with similar properties as our limiting domain depicted in Fig. 2.

In [30], Jimbo and Morita made a detailed study of the first k eigenvalues of the Laplace operator with Neumann boundary conditions in a domain $\Omega \subset \mathbb{R}^n$, which consists of exactly k fixed subdomains joined by thin channels. This k eigenvalues approach zero and the $k+1$ eigenvalue is uniformly bounded away from zero. The thickness of each channel is controlled by a small parameter $\epsilon > 0$ and these channels approach a line segment connecting two subdomains in a certain sense (some of these channels may be empty). With the characterization of the firsts k eigenvalues and eigenfunctions for the operator $-\Delta$ in this domain, in [29], the same authors apply the invariant manifold theory to show that the dynamics of an associated reaction–diffusion problem with a nonlinearity such that its Lipschitz constant is small (compared in some concrete way to the $k+1$ eigenvalue), is equivalent to the dynamics of a system of coupled ordinary differential equation in the invariant manifold. The fact that the Lipschitz constant of the nonlinearity is small prevents, in particular, any contribution to the dynamics from the channel. We would also like to mention the work [36], which extended somehow the results of [18,29].

In [28], Jimbo states a result on the continuity in the norm of the supremum of the attractors $\mathcal{A}_\epsilon$ for semilinear parabolic problems in dumbbell-type domains where the channel connecting the two disjoint domains is a straight cylindrical one. But no proofs are given.

In [41] the author develops a functional framework to treat nonlinear elliptic problems on sequences of domains $\{\Omega_n\}_{n=1}^\infty$. The sequence of domains is assumed to be nested, all of them contain the limit domain, Ω_0, and the sequence converges in measure to the limit domain. In this general context, the author obtains the upper semicontinuity of the set of equilibria. Moreover, under certain spectral convergence condition and certain restrictions on the Lipschitz constant of the nonlinearity, if the limit domain has a hyperbolic equilibrium, then for n large enough the equation has one and only one equilibrium nearby. The restriction on the Lipschitz constant of the nonlinearity is related to the restriction already mentioned in [30] and, in particular, it prevents from any contribution to the dynamics of the set $\Omega_n \setminus \Omega_0$. In particular, the results from this paper do not give information to the case of a dumbbell domain where the limit equation (1.2) has an equilibrium solution concentrated in the channel. This is the case, for example, if the channel is cylindrical and straight so that the operator $L(v) = v''$, the nonlinearity is $f(u) = k(u - u^3)$ and $k-1$ is larger than the first eigenvalue of the Laplace operator with Dirichlet boundary conditions in the segment R_0.

For the formation of patterns in nonconvex domains for reaction–diffusion equations with nonlinear boundary conditions, we refer to [12,13].

To the best of our knowledge the complete dynamical problem of a reaction–diffusion equation like (1.1) in a dumbbell domain in $\mathbb{R}^N$ with $N \geqslant 2$, with the following characteristics:

(1) the channel is not necessarily cylindrical,
(2) there is no restriction in the Lipschitz constant of the nonlinearities, and
(3) the limit equation (1.2) may have some dynamics in R_0, the limit of the thin channel,

has not been completely solved.

It is the purpose of this paper and of its continuation [7], to address this problem in its full generality.

This paper is organized as follows. In Section 2 we give the rigorous definition of the dumbbell domain, introduce some notation and state the main results of the paper, that is, the continuity of the set of equilibria and of its linear unstable manifold, Theorems 2.3 and 2.5. In this section we also state the basic result on the convergence of the resolvent of the linear operators, Proposition 2.7. In Section 3 we establish basic properties of the linear operators A_ϵ and A_0. Section 4 is devoted to the abstract results using the notion of compact convergence that, in particular, will lead to the continuity of eigenvalues and eigenfunctions of the linear operators involved in the equations. The continuity of equilibrium solutions in a general setting is addressed in Section 5. We have also included, in Sections 4 and 5, several examples that show how we apply this general results to the case of the dumbbell domain. We give a proof of Theorems 2.3 and 2.5 in Section 6. Finally, we have included Appendix A, which is devoted to the proof of compact convergence of the resolvent in the case of dumbbell-type domains, in particular, we show Proposition 2.7.

2. Definition of the domain and main results

Before we can state in a precise way our main result, let us define the perturbation of the domain we are considering.

The family of domains we are dealing with is the so-called dumbbell domain which, roughly speaking, consists of a pair of two fixed domains, Ω, joined by a thin channel R_ϵ which approaches a line segment as the parameter ϵ approaches zero. In order to describe such domains we need to introduce some terminology.

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a fixed open bounded and smooth domain such that there is an $l > 0$ satisfying

$$\Omega \cap \{(s, x'):\ s^2 + |x'|^2 < l^2\} = \{(s, x'):\ s^2 + |x'|^2 < l^2,\ s < 0\},$$
$$\Omega \cap \{(s, x'):\ (s-1)^2 + |x'|^2 < l^2\} = \{(s, x'):\ (s-1)^2 + |x'|^2 < l^2,\ s > 1\},$$
$$\Omega \cap \{(s, x'):\ 0 < s < 1,\ |x'| < l\} = \emptyset,$$

with $\{(0, x'):\ |x'| < l\} \cup \{(1, x'):\ |x'| < l\} \subset \partial\Omega$. We are using the standard notation $\mathbb{R}^N \ni x = (s, x')$, with $s \in \mathbb{R}$, $x' = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$.

The channel that we consider will be defined as $R_\epsilon = \{(s, \epsilon x'):\ (s, x') \in R_1\}$ and R_1 is defined as

$$R_1 = \{(s, x'):\ 0 \leqslant s \leqslant 1,\ x' \in \Gamma_1^s\}$$

and for all $0 \leqslant s \leqslant 1$, Γ_1^s is diffeomorphic to the unit ball in $\mathbb{R}^{N-1}$. That is, we assume that for each $s \in [0, 1]$, there exists a C^1 dipheomorphism

$$L_s : B(0, 1) \to \Gamma_1^s. \tag{2.1}$$

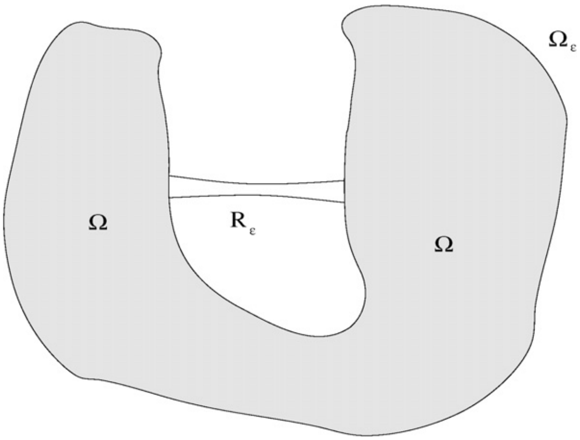

Fig. 3. Dumbbell domain with a connected Ω.

Moreover, if we define

$$L: \quad (0,1) \times B(0,1) \to R_1, \quad (s,z) \to \big(s, L_s(z)\big) \tag{2.2}$$

then L is a C^1 dipheomorphism.

Denote by $g(s) := |\Gamma_1^s|$ the $(N-1)$-dimensional Lebesgue measure of the set Γ_1^s. From the smoothness of R_1, we may assume that g is a smooth function defined in [0, 1]. In particular, there exist $d_0, d_1 > 0$ such that $d_0 \leqslant g(s) \leqslant d_1$ for all $s \in [0,1]$. Moreover, the channel R_ϵ collapses to the line segment $R_0 = \{(s,0)\colon 0 \leqslant s \leqslant 1\}$.

Remark 2.1. A very important class of channels will be those whose transversal sections Γ_1^s are disks centered at the origin of radius $r(s)$, that is

$$R_1 = \big\{(s,x')\colon |x'| < r(s),\ 0 \leqslant s \leqslant 1\big\}.$$

For this channel, $g(s) = \omega_{N-1} r(s)^{N-1}$ where ω_{N-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.

Many of the results in the literature are obtained for this particular channel, even for the completely straight channel given by $r(s) \equiv 1$, see, for instance, [22–25].

The dumbbell domain will be the domain $\Omega_\epsilon = \Omega \cup R_\epsilon$ for $\epsilon \in (0,1]$. Observe that we did not specify any connectedness property for Ω. Therefore, we can have the situation described in Fig. 1 or as in Fig. 3.

Consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u + u = f(u), & x \in \Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \tag{2.3}$$

defined in the dumbbell domain Ω_ϵ with f satisfying the following conditions:

(i) $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$ function,
(ii) $|f(u)| + |f'(u)| + |f''(u)| \leqslant C_1$ for all $u \in \mathbb{R}$.

Remark 2.2. Condition (ii) on the nonlinearity does not represent any restriction. Since the nonlinearity is assumed dissipative, we have L^∞ estimates of the attractors of the system and these estimates are uniform in the parameter ϵ. In particular, all solutions of (2.3) are bounded with a bound independent of ϵ. In case (ii) is not satisfied we can cut off the nonlinearity without modifying the solutions of the equation so that (ii) is satisfied.

We will denote by $\{\mathcal{E}_\epsilon\}_{\epsilon\in(0,1]}$ the set of solutions of problem (2.3). Under the above assumptions on the nonlinearity f, the set $\mathcal{E}_\epsilon$ is bounded in $L^\infty(\Omega_\epsilon)$, uniformly for $\epsilon \in (0,1]$.

The limit problem of (2.3) as $\epsilon \to 0$ is the following

$$\begin{cases} -\Delta w + w = f(w), & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ -\frac{1}{g}(g v_s)_s + v = f(v), & s \in (0,1), \\ v(0) = w(0), & v(1) = w(1). \end{cases} \tag{2.4}$$

Observe that a solution of the limit equation has two components, (w, v). The first one lives in Ω and the second one lives in $(0,1)$ or equivalently in the segment R_0. Moreover, the problem is not decoupled but it is interesting to note that it is coupled only in one direction. By this we mean that the function w is independent of v but v depends on w. Hence, to solve (2.4) we first find a solution w of the nonlinear problem in Ω,

$$\begin{cases} -\Delta w + w = f(w), & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Any solution of (2.5) is very smooth. In particular, it is in $C^0(\overline{\Omega})$ and it makes sense to take the trace of w at p_0 and p_1. Once this is obtained, we solve the nonlinear problem in the interval $(0,1)$ given by

$$\begin{cases} -\frac{1}{g}(g v_s)_s + v = f(v), & s \in (0,1), \\ v(0) = w(0), & v(1) = w(1). \end{cases} \tag{2.6}$$

The aim of this paper is to compare the solutions of the perturbed problem (2.3) and the solutions of the limit problem (2.4). Since the solutions live in different spaces we need to devise a mechanism to compare functions defined in the limiting domain $\Omega \cup R_0$ and in Ω_ϵ. First of all, we need an extension operator that maps functions (w, v) defined in $\Omega \cup R_0$ into functions defined in Ω_ϵ. The natural way to define this operator is to extend the functions defined in R_0 (that depend only on the variable s) constantly in the other $N-1$ variables, that is:

$$E_\epsilon(w, v)(x) = \begin{cases} w(x), & x \in \Omega, \\ v(s), & x = (s, y) \in R_\epsilon. \end{cases}$$

If we consider now X_ϵ, $0 \leqslant \epsilon \leqslant 1$, a family of functional spaces in Ω_ϵ (say, for instance, $X_\epsilon = L^2(\Omega_\epsilon)$, $0 < \epsilon \leqslant 1$ and $X_0 = L^2(\Omega) \times L^2(R_0)$), we can give the following definition of convergence: $u_\epsilon \to u_0$ if $\|u_\epsilon - E_\epsilon u_0\|_{X_\epsilon} \to 0$. This notion of convergence will strongly depend not only on the space X_ϵ but also, in a crucial way, on the metric chosen in X_ϵ. For instance, if we choose $X_\epsilon = L^2(\Omega_\epsilon)$ with the usual metric $\|u_\epsilon\|^2_{L^2(\Omega_\epsilon)} = \int_{\Omega_\epsilon} |u_\epsilon|^2$, we will have that the family of functions $u_\epsilon \equiv 1$ will converge to any function $u_0 \in L^2(\Omega) \times L^2(R_0)$ such that $u_0 = 1$ in Ω and it is arbitrary in R_0. In particular, with this choice of metric in $L^2(\Omega_\epsilon)$ the limit is not unique. On the other hand, if we define the metric in $L^2(\Omega_\epsilon)$ by

$$\|u_\epsilon\|^2_{L^2(\Omega_\epsilon)} = \int_\Omega |u_\epsilon|^2 + \frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} |u_\epsilon|^2$$

we are magnifying the functions in the channel R_ϵ with a factor $\epsilon^{-(N-1)}$. It is not difficult to show that with this definition, we have that the limit is unique. In particular, the functions $u_\epsilon \equiv 1$ converge to the function $u_0 \equiv 1$ in $\Omega \cup R_0$.

This considerations motivate the definition of the following spaces: for $1 \leqslant p < \infty$, the space U^p_ϵ is the space $L^p(\Omega_\epsilon)$ with the norm

$$\|\cdot\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|\cdot\|_{L^p(R_\epsilon)}$$

and denote by $U^p_0 = L^p(\Omega) \oplus L^p_g(0,1)$ where $L^p_g(0,1)$ is the space $L^p(0,1)$ with the norm

$$\|u\|_{L^p_g(0,1)} = \left(\int_0^1 g(s)\big|u(s)\big|^p\, ds \right)^{\frac{1}{p}}.$$

As a matter of fact, with the norm defined in Ω_ϵ we capture the behavior of the functions in the channel R_ϵ. Note that a function u defined in Ω_ϵ but independent of the y coordinate in the channel R_ϵ will satisfy

$$\|u\|_{U^p_\epsilon} = \|u\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|u\|_{L^p(R_\epsilon)} = \|u\|_{L^p(\Omega)} + \left(\int_0^1 g(s) u(s)^p\, ds \right)^{\frac{1}{p}}.$$

Notice that the extension operator E_ϵ maps U^p_0 to U^p_ϵ. Moreover,

$$\big\|E_\epsilon(w,v)\big\|_{U^p_\epsilon} = \big\|(w,v)\big\|_{U^p_0}.$$

We will also consider the spaces $H^1_\epsilon = H^1(\Omega) \oplus H^1(R_\epsilon)$ with the norm

$$\|\cdot\|_{H^1_\epsilon} = \|\cdot\|_{H^1(\Omega)} + \epsilon^{\frac{1-N}{2}} \|\cdot\|_{H^1(R_\epsilon)}.$$

With all this notation in mind we can state one of the main result in this paper.

Theorem 2.3. *Let $p > N$. If we denote by $\mathcal{E}_\epsilon$ the set of solutions of problem* (2.3) *for $\epsilon \in (0,1]$ and by $\mathcal{E}_0$ the set of solutions of problem* (2.4) *then we have the following*:

(i) *For any sequence $u^*_\epsilon \in \mathcal{E}_\epsilon$ with $\epsilon \to 0$, there exists a subsequence, that we also denote by ϵ, and a $u^*_0 \in \mathcal{E}_0$ such that*

$$\left\| u_\epsilon^* - E_\epsilon\left(u_0^*\right)\right\|_{U_\epsilon^p} \to 0, \quad as\ \epsilon \to 0, \tag{2.7}$$

$$\left\| u_\epsilon^* - E_\epsilon\left(u_0^*\right)\right\|_{H_\epsilon^1} \to 0, \quad as\ \epsilon \to 0. \tag{2.8}$$

Moreover, there exists an $\alpha \in (0,1)$ *such that for any compact set* $K \subset \overline{\Omega} \setminus \{p_0, p_1\}$ *we have* $\|u_\epsilon^* - u_0^*\|_{C^{1,\alpha}(K)} \to 0$ *as* $\epsilon \to 0$.

(ii) *For any hyperbolic equilibrium point* $u_0^* \in \mathcal{E}_0$, *there exists* $\eta > 0$ *and* $\epsilon_0 > 0$ *such that there exists one and only one equilibrium* u_ϵ^* *of* (2.3) *such that*

$$\left\| u_\epsilon^* - E_\epsilon\left(u_0^*\right)\right\|_{U_\epsilon^p} \leqslant \eta \quad for\ 0 < \epsilon \leqslant \epsilon_0. \tag{2.9}$$

Moreover, (2.7) *and* (2.8) *are satisfied.*

In particular, if every equilibrium of the limit problem is hyperbolic, then we have only a finite number of them, that is, $\mathcal{E}_0 = \{u_0^1, \dots, u_0^m\}$ *and there exists an* $\epsilon_0 > 0$ *such that* $\mathcal{E}_\epsilon = \{u_\epsilon^1, \dots, u_\epsilon^m\}$ *and* $u_\epsilon^i \to u_0^i$ *in the sense of* (2.7) *and* (2.8). *Moreover, the number of equilibria, m, is an odd number.*

Remark 2.4. An equilibrium point $u_0^* = (w_0, v_0) \in \mathcal{E}_0$ is hyperbolic if the linearization of (2.4) does not have any eigenvalue in the imaginary axis. Observe that λ is an eigenvalue of the linearization if we have solutions (ϕ, ψ) not identically zero, such that

$$\begin{cases} -\Delta\phi + \phi - f'(w_0)\phi = \lambda\phi, & x \in \Omega, \\ \frac{\partial\phi}{\partial n} = 0, \quad x \in \partial\Omega, \\ -\frac{1}{g}(g\psi_s)_s + \psi - f'(v_0)\psi = \lambda\psi, & s \in (0,1), \\ \psi(0) = \phi(0), \qquad \psi(1) = \phi(1). \end{cases} \tag{2.10}$$

From (2.10) it is easy to see that all eigenvalues are real (although the operator obtained through linearization is not self-adjoint) and that $\lambda = 0$ is not an eigenvalue of (2.10) if $\lambda = 0$ is not an eigenvalue of the operator $-\Delta\phi + \phi - f'(w_0)\phi = \lambda\phi$ in Ω with homogeneous Neumann boundary condition nor an eigenvalue of the operator $-\frac{1}{g}(g\psi_s)_s + \psi - f'(v_0)\psi = \lambda\psi$ in $(0,1)$ with homogeneous Dirichlet boundary conditions.

As a matter of fact we will be able to obtain more information on the relation between the linearized operators around equilibria. We will show the following theorem.

Theorem 2.5. *In the conditions of Theorem* 2.3 *let* u_ϵ^* *be a sequence of equilibria of* (2.3) *and* $u_0^* = (w_0, v_0)$ *an equilibrium of* (2.4) *satisfying* (2.7) *and* (2.8). *Denote by* $\{\lambda_n^\epsilon\}_{n=1}^\infty$ *the set of eigenvalues* (*ordered and counting multiplicity*) *of the linearization around* u_ϵ^*, *that is, the eigenvalues of*

$$\begin{cases} -\Delta\phi_\epsilon + \phi_\epsilon - f'(u_\epsilon^*)\phi_\epsilon = \lambda\phi_\epsilon, & x \in \Omega_\epsilon, \\ \frac{\partial\phi_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \tag{2.11}$$

and by $\{\phi_n^\epsilon\}_{n=1}^\infty$ *a corresponding set of orthonormal eigenfunctions.*

Also, denote by $\{\lambda_n^0\}_{n=1}^\infty$ *the set of eigenvalues of* (2.10)*, ordered and counting its algebraic multiplicity, and denote by* $\{\phi_n^0\}_{n=1}^\infty$ *a corresponding set of generalized eigenfunctions. Then, we have*

$$\lambda_n^\epsilon \xrightarrow{\epsilon\to 0} \lambda_n^0 \quad \textit{for all } n = 1, 2, \ldots.$$

Also if n *is such that* $\lambda_n^0 < \lambda_{n+1}^0$ *and we define*

$$W_n^0 = span\left[\phi_1^0, \ldots, \phi_n^0\right], \qquad W_n^\epsilon = span\left[\phi_1^\epsilon, \ldots, \phi_n^\epsilon\right],$$

then

$$dist_{U_\epsilon^p}\left(W_n^\epsilon, E_\epsilon W_n^0\right) \xrightarrow{\epsilon\to 0} 0, \qquad dist_{H_\epsilon^1}\left(W_n^\epsilon, E_\epsilon W_n^0\right) \xrightarrow{\epsilon\to 0} 0. \tag{2.12}$$

In particular, if u_0^* *is a hyperbolic equilibrium point of the limit equation and* u_ϵ^* *is the sequence of equilibrium points such that* $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon^p} \xrightarrow{\epsilon\to 0} 0$ *given by Theorem* 2.3*, then for* ϵ *small enough,* u_ϵ^* *is also hyperbolic and its linearized unstable manifold converge, in the sense of* (2.12)*, to the linearized unstable manifold of* u_0^**. In particular, the dimension of the unstable manifolds of* u_ϵ^* *and of* u_0^* *coincide.*

Remark 2.6. (i) In relation to (2.12), the distance of two subspaces is the symmetric Hausdorff distance of the unit balls of the two subspaces, that is, if W_1, W_2 are subspaces of the Banach space U, then

$$dist_U(W_1, W_2) = \sup_{x\in B_{W_1}} \inf_{y\in B_{W_2}} \|x-y\|_U + \sup_{y\in B_{W_2}} \inf_{x\in B_{W_1}} \|x-y\|_U,$$

where B_{W_1} and B_{W_2} are the unit balls of W_1 and W_2, respectively.

(ii) The convergence of the linearized unstable manifold is a first step needed to prove the convergence of the attractors. As it is mentioned in the introduction, this result will be accomplished in [7].

The results of the above theorems will be obtained after a careful analysis on the behavior of the resolvent of the linear operators is performed. Actually, we will prove the following basic and important result:

Proposition 2.7. *For* $f_\epsilon \in U_\epsilon^p$, $\epsilon \in (0, 1]$*, let* u_ϵ *be the solution of*

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon, & x \in \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \tag{2.13}$$

and for $(f, h) \in U_0^p$ *let* (w, v) *be the solution of*

$$\begin{cases} -\Delta w + w = f, \quad x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega, \\ -\frac{1}{g}(g v_s)_s + v = h, \quad s \in (0, 1), \\ v(0) = w(0), \qquad v(1) = w(1). \end{cases} \tag{2.14}$$

Then, we have

(1) *With $p > N/2$, there exists a constant $C > 0$, independent of ϵ and of f_ϵ, such that*

$$\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\|f_\epsilon\|_{U_\epsilon^p}.$$

(2) *With $p > N$, if $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$, $\epsilon \in (0,1]$, there is a subsequence, denoted by ϵ again and $(f,h) \in U_0^p$, such that if (w,v) are given by* (2.14) *then the following holds*:

$$\text{(i)} \quad \big\|u_\epsilon - E_\epsilon(w,v)\big\|_{H_\epsilon^1} \xrightarrow{\epsilon\to 0} 0,$$

$$\text{(ii)} \quad \big\|u_\epsilon - E_\epsilon(w,v)\big\|_{U_\epsilon^q} \xrightarrow{\epsilon\to 0} 0, \quad \textit{for all } 1 \leqslant q < \infty,$$

$$\text{(iii)} \quad \|u_\epsilon - w\|_{C^{1,\alpha}(K)} \xrightarrow{\epsilon\to 0} 0, \quad \textit{for all compact } K \subset \overline{\Omega} \setminus \{p_0, p_1\}.$$

(3) *With $p > N$, if we have $\|f_\epsilon - E_\epsilon(f,h)\|_{U_\epsilon^p} \xrightarrow{\epsilon\to 0} 0$ then we have* (i)–(iii) *for the whole sequence.*

The proof of this proposition is written in Appendix A.

3. Problems (2.3) and (2.4)

We will write both problems, (2.3) and (2.4) as abstract problems in the Banach spaces U_ϵ^p and U_p^0, respectively.

Since for fixed ϵ, the space U_ϵ^p is equivalent to $L^p(\Omega_\epsilon)$, problem (2.3) can be written as an abstract equation of semilinear type of the form

$$A_\epsilon u = F_\epsilon(u), \tag{3.1}$$

where $A_\epsilon : \mathcal{D}(A_\epsilon) \subset U_\epsilon^p \to U_\epsilon^p$, $1 \leqslant p < \infty$, is the linear operator defined by

$$\mathcal{D}(A_\epsilon) = \big\{u \in W^{2,p}(\Omega_\epsilon)\colon\ \Delta u \in U_\epsilon^p,\ \partial u/\partial n = 0 \text{ in } \partial\Omega_\epsilon\big\},$$
$$A_\epsilon u = -\Delta u + u, \quad u \in \mathcal{D}(A_\epsilon) \tag{3.2}$$

and the nonlinearity $F_\epsilon : U_\epsilon \to U_\epsilon$ is the Nemitskĭi operator generated by f, that is $F_\epsilon(u_\epsilon)(x) = f(u_\epsilon(x))$.

The operator A_ϵ is sectorial and the following estimate holds

$$\big\|(A_\epsilon + \lambda)^{-1}\big\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \leqslant \frac{C}{|\lambda|}, \quad \text{for } \lambda \in \Sigma_\theta, \tag{3.3}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C}\colon\ |\arg(\lambda - 1)| \leqslant \theta\}$, $\theta > \frac{\pi}{2}$, and C is a constant that does not depend on ϵ. This follows from the fact that the localization of the numerical range in the complex plane can be done independently of ϵ, see [37, p. 215].

Define the limit linear operator, $A_0 : \mathcal{D}(A_0) \subset U_0^p \to U_0^p$, as

$$A_0(w, v) = \left(-\Delta w + w, -\frac{1}{g}(g v_x)_x + v\right), \quad (w, v) \in \mathcal{D}(A_0), \tag{3.4}$$

with domain

$$\mathcal{D}(A_0) = \left\{(w, v) \in U_0^p \colon\ w \in \mathcal{D}\left(\Delta_N^\Omega\right),\ (g v_x)_x \in L_g^p(0, 1),\ v(0) = w(0),\ v(1) = w(1)\right\}, \tag{3.5}$$

where Δ_N^Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$.

We have the following proposition:

Proposition 3.1. *The operator A_0 defined by* (3.4) *has the following properties*:

(i) *$\mathcal{D}(A_0)$ is dense in U_0^p,*
(ii) *If $p > N/2$ then A_0 is a closed operator,*
(iii) *A_0 has compact resolvent.*

Proof. (i) Let $(w, v) \in L^p(\Omega) \oplus L_g^p(0, 1)$. Let $(w_n, v_n) \in C_0^\infty(\Omega) \oplus C_0^\infty(0, 1)$ with $(w_n, v_n) \to (w, v)$ in $L^p(\Omega) \oplus L_g^p(0, 1)$, then $(w_n, v_n) \in \mathcal{D}(A_0)$ and the result is proved.

(ii) Let $(w_n, v_n) \in \mathcal{D}(A_0)$ be such that $(w_n, v_n) \to (w, v)$ and $A_0(w_n, v_n) \to (\phi, \psi)$ in $L^p(\Omega) \oplus L_g^p(0, 1)$. Since $w_n \in \mathcal{D}(\Delta_N^\Omega)$ and Δ_N^Ω is a closed operator in $L^p(\Omega)$, see [20], we have that $w \in \mathcal{D}(\Delta_N^\Omega)$ and $w_n \to w$ in $W^{2,p}(\Omega)$. In particular, $-\Delta w_n \to -\Delta w$ and since $p > N/2$ we have $W^{2,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, which implies that $w_n(0) \to w(0)$ and $w_n(1) \to w(1)$. On the other hand, $v_n \to v$ and $\psi_n = -\frac{1}{g}(g v_n')' + v_n \to \psi$ in $L_g^p(0, 1)$. Now

$$\begin{cases} -\frac{1}{g}(g v_n')' + v_n = \psi_n, & s \in (0, 1), \\ v_n(0) = w_n(0), & v_n(1) = w_n(1). \end{cases}$$

Making the change of variables $z_n = v_n - \xi_n$, where ξ_n is the solution of the following problem

$$\begin{cases} -\frac{1}{g}(g \xi_n')' = 0, & s \in (0, 1), \\ \xi_n(0) = w_n(0), & \xi_n(1) = w_n(1), \end{cases} \tag{3.6}$$

we have

$$\begin{cases} -\frac{1}{g}(g z_n')' = \psi_n - v_n, & s \in (0, 1), \\ z_n(0) = z_n(1) = 0. \end{cases}$$

It is easy to see that

$$\xi_n(s) = w_n(0) + \frac{w_n(1) - w_n(0)}{\int_0^1 \frac{1}{g(\theta)}\, d\theta} \int_0^s \frac{1}{g(\theta)}\, d\theta \tag{3.7}$$

and, since $w_n(0) \to w(0)$, $w_n(1) \to w(1)$, it follows that $\xi_n \to \xi$, where ξ is the solution of the following problem:

$$\begin{cases} -\frac{1}{g}(g\xi')' = 0, & s \in (0,1), \\ \xi(0) = w(0), & \xi(1) = w(1). \end{cases} \tag{3.8}$$

Moreover, since the operator $L(v) = -\frac{1}{g}(gv')'$ with homogeneous Dirichlet boundary conditions at $s = 0$ and $s = 1$ is closed in $L^p_g(0,1)$, we have that $z_n \to z$ in $L^p_g(0,1)$ where z satisfies

$$\begin{cases} -\frac{1}{g}(gz')' = \psi - v, & s \in (0,1), \\ z(0) = z(1) = 0. \end{cases}$$

From which it follows that $v_n = z_n + \xi_n \to z + \xi = v$, and v satisfies

$$\begin{cases} -\frac{1}{g}(gv')' + v = \psi, & s \in (0,1), \\ v(0) = w(0), & v(1) = w(1). \end{cases} \tag{3.9}$$

(iii) Since $\mathcal{D}(A_0) \subset W^{2,p}(\Omega) \oplus W^{1,p}_g(0,1) \hookrightarrow L^p(\Omega) \oplus L^p_g(0,1)$ and since the embedding $W^{2,p}(\Omega) \oplus W^{1,p}_g(0,1) \hookrightarrow L^p(\Omega) \oplus L^p_g(0,1)$ is compact, it follows that A_0 has compact resolvent. $\square$

Remark 3.2. Even though Proposition 3.1 states several important properties of the operator A_0, we would like to mention that A_0 is not a sectorial operator. Its spectrum is all real and, therefore, it is contained in a sector but the required resolvent estimate

$$\left\|(A_0 + \lambda I)^{-1}\right\|_{\mathcal{L}(U_0^p)} \leqslant \frac{C}{|\lambda + a|}$$

is not satisfied. To see this, we refer to [7].

4. Abstract compact convergence results

In this section we develop (following [9]) the basic abstract tool that we are going to use to compare two linear problems defined in different spaces. This theory will be applied to compare the linear problem defined in the dumbbell domain Ω_ϵ with the linear problem defined in the limit domain $\Omega \cup R_0$. This will be illustrated throughout several examples included in the section.

Hence, let U_ϵ be a family of Banach spaces for $\epsilon \in [0,1]$ and assume there is a family of linear operators $E_\epsilon : U_0 \to U_\epsilon$ with the property that

$$\|E_\epsilon u\|_{U_\epsilon} \xrightarrow{\epsilon \to 0} \|u\|_{U_0}, \quad \text{for all } u \in U_0. \tag{4.1}$$

Example 4.1. Let $\Omega_\epsilon = \Omega \cup R_\epsilon$ be the dumbbell domain defined in Section 2 and let U^p_ϵ and U^p_0 be the spaces defined also in Section 2. Consider the extension operators $E_\epsilon : U^p_0 \to U^p_\epsilon$ as

$$E_\epsilon(w,v)(x) = \begin{cases} w(x), & x \in \Omega, \\ v(s), & x = (s,y) \in R_\epsilon. \end{cases}$$

It is very easy to verify that $\|E_\epsilon(w,v)\|_{U^p_\epsilon} = \|(u,v)\|_{U^p_0}$.

Definition 4.2. We say that a sequence $\{u_\epsilon\}_{\epsilon\in(0,1]}$ E-converges to u if $\|u_\epsilon - E_\epsilon u\|_{U_\epsilon} \xrightarrow{\epsilon\to 0} 0$. We write this as $u_\epsilon \xrightarrow{E} u$.

With this notion of convergence we introduce the notion of compactness.

Definition 4.3. A sequence $\{u_n\}_{n\in\mathbb{N}}$, with $u_n \in U_{\epsilon_n}$ and $\epsilon_n \to 0$, is said pre-compact if for each subsequence $\{u_{n'}\}$ there is another subsequence $\{u_{n''}\}$ and an element $u \in U_0$ such that $u_{\epsilon_{n''}} \xrightarrow{E} u$. The family $\{u_\epsilon\}_{\epsilon\in(0,1]}$ is said pre-compact if each sequence $\{u_{\epsilon_n}\}$, with $\epsilon_n \to 0$, is precompact.

Definition 4.4. We say that a family of operators $\{B_\epsilon \in \mathcal{L}(U_\epsilon)\colon \epsilon \in (0,1]\}$ converges to $B_0 \in \mathcal{L}(U_0)$, as $\epsilon \to 0$, if $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$, whenever $u_\epsilon \xrightarrow{E} u \in U_0$. We write $B_\epsilon \xrightarrow{EE} B_0$.

Definition 4.5. We say that a family of compact operators $\{B_\epsilon \in \mathcal{L}(U_\epsilon)\colon \epsilon \in (0,1]\}$ converges compactly to a compact operator $B_0 \in \mathcal{L}(U_0)$ if for any family $\{u_\epsilon\}$ with $\|u_\epsilon\|_{U_\epsilon} = 1$, $\epsilon \in (0,1]$, the family $\{B_\epsilon u_\epsilon\}$ is relatively compact and, moreover, $B_\epsilon \xrightarrow{EE} B_0$. We denote this as $B_\epsilon \xrightarrow{CC} B_0$.

Example 4.6. Let Ω_ϵ, Ω_0, U_ϵ^p, U_0^p be the domains and the spaces of the dumbbell domain of Example 4.1. Let A_ϵ, A_0 be the operators defined in Section 3 and consider the operators $B_\epsilon \in \mathcal{L}(U_\epsilon^p)$ defined by $B_\epsilon = A_\epsilon^{-1}$, that is, $B_\epsilon f_\epsilon = u_\epsilon$ where u_ϵ is the solution of

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon, & x \in \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \tag{4.2}$$

and $B_0 \in \mathcal{L}(U_0^p)$ be the operator defined by $B_0 = A_0^{-1}$, that is, $B_0(f,h) = (u,v)$, where (u,v) is the solution of

$$\begin{cases} -\Delta w + w = f, & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ -\frac{1}{g(s)}(g(s)v'(s))' + v(s) = h(s), & s \in (0,1), \\ v(0) = w(0), & v(1) = w(1). \end{cases} \tag{4.3}$$

We will prove in Appendix A that if $p > N$, then $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$. This is the fundamental result that will give us the key to all the results of the paper. Also, notice that this is what Proposition 2.7 states.

The following lemma is a key result.

Lemma 4.7. *Assume that $\{B_\epsilon \in \mathcal{L}(U_\epsilon)\}_{\epsilon\in(0,1]}$ converges compactly to B_0 as $\epsilon \to 0$. Then,*

(i) $\|B_\epsilon\|_{\mathcal{L}(U_\epsilon)} \leqslant C$ *for some constant C, independent of ϵ.*
(ii) *Assume that $\mathcal{N}(I + B_0) = \{0\}$ then, there exists an $\epsilon_0 > 0$ and $M > 0$ such that*

$$\big\|(I + B_\epsilon)^{-1}\big\|_{\mathcal{L}(U_\epsilon)} \leqslant M, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{4.4}$$

Proof. (i) If the norms are not bounded, then we can choose a sequence of $\epsilon_n \to 0$ and $u_{\epsilon_n} \in U_{\epsilon_n}$ with $\|u_{\epsilon_n}\|_{U_{\epsilon_n}} = 1$ such that $\|B_{\epsilon_n} u_{\epsilon_n}\| \to +\infty$. But this is in contradiction with the compact convergence of B_ϵ given in Definition 4.5.

(ii) Because B_ϵ is compact for every $\epsilon \in [0,1]$, estimate (4.4) is equivalent, say, to

$$\left\|(I + B_\epsilon)u_\epsilon\right\|_{U_\epsilon} \geqslant \frac{1}{M}, \quad \forall \epsilon \in [0, \epsilon_0] \text{ and } \forall u_\epsilon \in U_\epsilon \text{ with } \|u_\epsilon\| = 1.$$

Suppose that this is not true; that is, suppose that there is a sequence $\{u_n\}$, with $u_n \in U_{\epsilon_n}$, $\|u_n\| = 1$ and $\epsilon_n \to 0$ such that $\|(I + B_{\epsilon_n})u_n\| \to 0$. Since $\{B_{\epsilon_n} u_n\}$ has a convergent subsequence, which we again denote by $\{B_{\epsilon_n} u_n\}$, to u, $\|u\| = 1$, then $u_n + B_{\epsilon_n} u_n \to 0$ and $u_n \to -u$. This implies that $(I + B)u = 0$ contradicting our hypothesis. □

In general, we will have that the operators B_ϵ are inverses of certain differential operators A_ϵ. Hence, assume we have operators $A_\epsilon : \mathcal{D}(A_\epsilon) \subset U_\epsilon \to U_\epsilon$ for $\epsilon \in [0,1]$ and assume that we have the following hypotheses:

$$\boxed{A_\epsilon \text{ is closed, has compact resolvent, } 0 \in \rho(A_\epsilon),\ \epsilon \in [0,1] \text{ and } A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}.} \tag{4.5}$$

Lemma 4.8. *Let A_ϵ be such that* (4.5) *hold. Then, for any $\lambda \in \rho(A_0)$, there is an $\epsilon_\lambda > 0$ such that $\lambda \in \rho(A_\epsilon)$ for all $\epsilon \in [0, \epsilon_\lambda]$ and there is a constant $M_\lambda > 0$ such that*

$$\left\|(\lambda - A_\epsilon)^{-1}\right\| \leqslant M_\lambda, \quad \forall \epsilon \in [0, \epsilon_\lambda]. \tag{4.6}$$

Furthermore, $(\lambda - A_\epsilon)^{-1}$ converges compactly to $(\lambda - A_0)^{-1}$ as $\epsilon \to 0$.

Proof. From (4.5) and since $\lambda \in \rho(A_0)$ it is easy to see that

$$(\lambda - A_0)^{-1} = -A_0^{-1}\left(I - \lambda A_0^{-1}\right)^{-1}.$$

Since $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$, applying Lemma 4.7(i) and (ii), we get that the operator $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1}$ is well defined and bounded. Easy computations show that actually $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1} = (\lambda - A_\epsilon)^{-1}$. Hence $\lambda \in \rho(A_\epsilon)$ and we obtain (4.6).

In order to show the compact convergence of $(\lambda - A_\epsilon)^{-1}$ to $(\lambda - A_0)^{-1}$ we proceed as follows. Since A_ϵ^{-1} converges compactly to A_0^{-1} and since $\{(I - \lambda A_\epsilon^{-1})\colon 0 \leqslant \epsilon \leqslant \epsilon_\lambda\}$ is bounded we conclude that:

- If $\|u_\epsilon\|_{U_\epsilon} = 1$ then $(\lambda - A_\epsilon)^{-1} u_\epsilon = -A_\epsilon^{-1} w_\epsilon$ with $w_\epsilon = (I - \lambda A_\epsilon^{-1})^{-1} u_\epsilon$ which is uniformly bounded in ϵ. Hence $(\lambda - A_\epsilon)^{-1} u_\epsilon$ has an E-convergent subsequence.
- If $u_\epsilon \xrightarrow{E} u$ then $A_\epsilon^{-1} u_\epsilon \xrightarrow{E} A_0^{-1} u$. Now, for any subsequence of $\{(\lambda - A_\epsilon)^{-1} u_\epsilon\}$ there is a subsequence (which we again denote by $\{(\lambda - A_\epsilon)^{-1} u_\epsilon\}$) and a y, such that

$$(\lambda - A_\epsilon)^{-1} u_\epsilon = -\left(I - \lambda A_\epsilon^{-1}\right)^{-1} A_\epsilon^{-1} u_\epsilon = z_\epsilon \xrightarrow{E} y.$$

Therefore,

$$A_0^{-1}u \xleftarrow{E} A_\epsilon^{-1}u_\epsilon = -\big(I - \lambda A_\epsilon^{-1}\big)z_\epsilon \xrightarrow{E} -\big(I - \lambda A_0^{-1}\big)y.$$

This implies that $y = (\lambda - A_0)^{-1}u$. In particular, y is independent of the subsequence chosen. This implies that the whole sequence $(\lambda - A_\epsilon)^{-1}u_\epsilon$ converges to $y = (\lambda - A_0)^{-1}u$. Thus, $(\lambda - A_\epsilon)^{-1} \xrightarrow{EE} (\lambda - A_0)^{-1}$.

From this we have the compact convergence of $(\lambda - A_\epsilon)^{-1} \xrightarrow{CC} (\lambda - A_0)^{-1}$ and the result is proved. □

Lemma 4.9. *If λ and δ are chosen such that $S_\delta := \{\mu \in \mathbb{C}\colon |\mu - \lambda| = \delta\}$ satisfies $\sigma(A_0) \cap S_\delta = \emptyset$ then, there exists $\epsilon_{S_\delta} > 0$ such that $\sigma(A_\epsilon) \cap S_\delta = \emptyset$ for all $\epsilon \leqslant \epsilon_{S_\delta}$.*

Proof. Suppose not. Then, there are sequences $\epsilon_n \to 0$, $\lambda_n \in S_\delta$ (which we may assume convergent to λ) and $u_{\epsilon_n} \in U_{\epsilon_n}$, $\|u_{\epsilon_n}\|_{U_{\epsilon_n}} = 1$ such that $u_{\epsilon_n} - (A_{\epsilon_n})^{-1}\lambda_n u_{\epsilon_n} = 0$ or equivalently $\lambda_n (A_{\epsilon_n})^{-1}u_{\epsilon_n} = u_{\epsilon_n}$. It follows from compact convergence that u_{ϵ_n} has a convergent subsequence to $u \in U_0$, $\|u\|_{U_0} = 1$ and that $A_0 u = \lambda u$ which contradicts our assumption. □

For an isolated point $\lambda \in \sigma(A_0)$ we associate its generalized eigenspace $W(\lambda, A_0) = Q(\lambda, A_0)U_0$, where

$$Q(\lambda, A_0) = \frac{1}{2\pi i} \int\limits_{|\xi - \lambda| = \delta} (\xi I - A_0)^{-1}\, d\xi,$$

and δ is chosen so small that there is no other point of $\sigma(A_0)$ in the disc $\{\xi \in \mathbb{C}\colon |\xi - \lambda| \leqslant \delta\}$. It follows from the previous lemma that there is ϵ_{S_δ} such that $\rho(A_\epsilon) \supset S_\delta$ for all $\epsilon \leqslant \epsilon_{S_\delta}$. We denote by $W(\lambda, A_\epsilon) = Q(\lambda, A_\epsilon)U_\epsilon$, where

$$Q(\lambda, A_\epsilon) = \frac{1}{2\pi i} \int\limits_{|\xi - \lambda| = \delta} (\xi I - A_\epsilon)^{-1}\, d\xi.$$

Our next result says that the spectrum of A_ϵ, for ϵ small, *approaches* the spectrum of A_0. We already know that the spectrum of A_ϵ or A_0 consists of isolated eigenvalues only.

Theorem 4.10. *Let A_ϵ, A_0 be such that* (4.5) *is satisfied. Then the following statements hold.*

- (i) *If $\lambda_0 \in \sigma(A_0)$, there exists a sequence $\epsilon_n \to 0$ and $\lambda_n \in \sigma(A_{\epsilon_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda_0$ as $n \to \infty$.*
- (ii) *If for some sequences $\epsilon_n \to 0$, $\lambda_n \in \sigma(A_{\epsilon_n})$, $n \in \mathbb{N}$, one has $\lambda_n \to \lambda_0$ as $n \to \infty$, then $\lambda_0 \in \sigma(A_0)$.*
- (iii) *There exists $\epsilon_0 > 0$ such that* $\dim W(\lambda_0, A_\epsilon) = \dim W(\lambda_0, A_0)$ *for all* $0 \leqslant \epsilon \leqslant \epsilon_0$.
- (iv) *If $u \in W(\lambda_0, A_0)$, there exists a sequence $\{u_\epsilon\}$, $u_\epsilon \in W(\lambda_0, A_\epsilon)$, such that $u_\epsilon \xrightarrow{E} u$.*
- (v) *If $\epsilon_n \to 0$, and $u_n \in W(\lambda, A_{\epsilon_n})$, satisfies $\|u_n\|_{U_{\epsilon_n}} = 1$ then, $\{u_n\}$ has an E-convergent subsequence and any limit point of this sequence belongs to $W(\lambda_0, A_0)$.*

Proof. (i) Let us take any $\lambda_0 \in \sigma(A_0)$ and consider $\mathcal{O}(\lambda_0, \delta) = \{\lambda \in \mathbb{C}\colon |\lambda - \lambda_0| \leqslant \delta\}$ such that $\mathcal{O}(\lambda_0, \delta) \cap \sigma(A_0) = \{\lambda_0\}$. To show that there is $\epsilon_0 > 0$ such that $\|(\lambda - A_\epsilon)^{-1}\| = O(1)$ for $\epsilon \in [0, \epsilon_0]$ and $\lambda \in \partial\mathcal{O}(\lambda_0, \delta)$ it is enough to prove that

$$\left\|\left(I - \lambda A_\epsilon^{-1}\right)^{-1}\right\| = O(1), \quad \epsilon \in [0, \epsilon_0],\ \lambda \in \partial\mathcal{O}(\lambda_0, \delta).$$

If that is not the case there will be a sequence $\lambda_n \in \partial\mathcal{O}(\lambda_0, \delta)$ (which we may assume convergent to some $\tilde{\lambda} \in \partial\mathcal{O}(\lambda_0, \delta)$), a sequence $u_n \in U_{\epsilon_n}$, $\|u_n\|_{U_{\epsilon_n}} = 1$, and a sequence $\epsilon_n \to 0$ such that

$$\left\|\left(I - \lambda_n (A_{\epsilon_n})^{-1}\right)u_n\right\|_{U_{\epsilon_n}} \overset{n\to\infty}{\longrightarrow} 0.$$

Since $\tilde{\lambda} \in \rho(A_0)$, that is in contradiction with Lemma 4.7.

Assume now that $\mathcal{O}(\lambda_0, \delta) \subset \rho(A_\epsilon)$. The function $(\lambda - A_\epsilon)^{-1}$ is holomorphic. From what we have just proved and from the Maximum Modulus Theorem one can see that

$$\left\|\left(I - \lambda_0 A_\epsilon^{-1}\right)^{-1}\right\| \leqslant \max_{|\lambda-\lambda_0|=\delta,\ \epsilon\in[0,\epsilon_0]} \left\|\left(I - \lambda A_\epsilon^{-1}\right)^{-1}\right\| = c < \infty.$$

Hence if $\epsilon_n \to 0$ and $u_n \to u$, it follows from Lemma 4.7 that

$$\left\|\left(\lambda_0 A_0^{-1} - I\right)u\right\|_{U_0} = \lim_{n\to\infty} \left\|\left(\lambda_0 A_{\epsilon_n}^{-1} - I\right)u_n\right\|_{U_{\epsilon_n}} \geqslant c\|u\|_{U_0},$$

for some $c > 0$ and $\lambda_0 \in \rho(A_0)$. So any $\mathcal{O}(\lambda_0, \delta)$ contains some point of $\sigma(A_\epsilon)$, for suitably small ϵ.

(ii) Assume now that $\epsilon_n \to 0$, $\{\lambda_n\}, \lambda_n \in \sigma(A_{\epsilon_n})$, is such that $\lambda_n \to \lambda$ and $\|(I - \lambda_n (A_{\epsilon_n})^{-1})u_n\| = 0$, $\|u_n\| = 1$. Then

$$\left\|\left(I - \lambda (A_{\epsilon_n})^{-1}\right)u_n\right\|_{U_{\epsilon_n}} = \left\|\left(I - \lambda_n (A_{\epsilon_n})^{-1}\right)u_n - (\lambda - \lambda_n)(A_{\epsilon_n})^{-1}u_n\right\|_{U_{\epsilon_n}} \to 0$$

as $n \to \infty$. Once $\|u_n\| = 1$ we have, taking subsequences if necessary, $(A_{\epsilon_n})^{-1}u_n \to y$ and $u_n \to u$, $\|u\| = 1$. Therefore $u - \lambda A_0^{-1}u = 0$, $u \neq 0$, which means $\lambda \in \sigma(A_0)$.

(iii) Since $(\lambda - A_\epsilon)^{-1} \overset{\epsilon\to 0}{\longrightarrow} (\lambda - A_0)^{-1}$ for any λ such that $|\lambda - \lambda_0| = \delta$ and since

$$\left\{\left\|(\lambda - A_\epsilon)^{-1}\right\|\colon 0 \leqslant \epsilon \leqslant \epsilon_0\right\}$$

is bounded, it follows from Dominated Convergence Theorem that $Q_\epsilon(\lambda_0) \overset{\epsilon\to 0}{\longrightarrow} Q(\lambda_0)$.

If $v_1, \ldots, v_k$ is a basis for $W(\lambda_0, A_0) = Q_0(\lambda_0)U_0$, it is easy to see that, for suitably small ϵ,

$$\left\{Q_\epsilon(\lambda_0)E_\epsilon v_1, \ldots, Q_\epsilon(\lambda_0)E_\epsilon v_k\right\}$$

is a linearly independent set in $Q_\epsilon(\lambda_0)U_\epsilon$. Hence $\operatorname{rank}(Q_\epsilon(\lambda_0)) \geqslant \operatorname{rank}(Q(\lambda_0))$.

We prove the converse inequality assuming that $Q_\epsilon(\lambda_0) \to Q(\lambda_0)$ compactly. If for some sequence $\epsilon_n \to 0$, $\operatorname{rank}(Q_{\epsilon_n}(\lambda_0)) > \operatorname{rank}(Q(\lambda_0))$, it follows from Lemma IV.2.3 in [31] that, for each $n \in \mathbb{N}$, there is a $u_n \in W(\lambda_0, A_{\epsilon_n})$, $\|u_n\| = 1$, such that $dist(u_n, W(\lambda_0, A_0)) = 1$. From the compact convergence we can assume that $Q_{\epsilon_n}(\lambda_0)u_n = u_n \to Q_0(\lambda_0)u_0 = u_0$, hence

$$1 \leqslant \left\|u_n - Q_0(\lambda_0)u_n\right\| = \left\|Q_{\epsilon_n}(\lambda_0)u_n - Q_0(\lambda_0)u_n\right\| \to 0.$$

So we need to prove just compact convergence $Q_\epsilon(\lambda_0) \to Q(\lambda_0)$ and that follows from the compact convergence of $A_\epsilon^{-1} \to A_0^{-1}$, from the uniform boundedness of $\|(\zeta A_\epsilon^{-1} - I)^{-1}\|$ ($|\zeta - \lambda_0| = \delta$ and $\epsilon \in [0, \epsilon_0]$), given by Lemma 4.7, and from the formula

$$Q_\epsilon(\lambda_0) = \frac{1}{2\pi i} \int\limits_{|\zeta-\lambda_0|=\delta} (\zeta I - A_\epsilon)^{-1}\, d\zeta = A_\epsilon^{-1} \frac{1}{2\pi i} \int\limits_{|\zeta-\lambda_0|=\delta} \left(\zeta A_\epsilon^{-1} - I\right)^{-1} d\zeta.$$

(iv) This follows taking $u_\epsilon = Q_\epsilon(\lambda_0) E_\epsilon u$.

(v) Follows from the compact convergence of Q_ϵ to Q proved in (iii). □

Proposition 4.11. *Let A_ϵ, A_0 be such that condition* (4.5) *is satisfied. Let K be a compact subset of $\rho(A_0)$. Then, there is a constant $\epsilon_K > 0$ such that $K \subset \rho(A_\epsilon)$ for all $\epsilon \in [0, \epsilon_K]$ and*

$$\sup_{\lambda \in K,\, \epsilon \in [0,\epsilon_K]} \left\| (\lambda - A_\epsilon)^{-1} \right\| < \infty. \tag{4.7}$$

Furthermore, for any $u \in U_0$

$$\sup_{\lambda \in K} \left\| (\lambda - A_\epsilon)^{-1} E_\epsilon u - E_\epsilon (\lambda - A_0)^{-1} u \right\|_{U_\epsilon} \xrightarrow{\epsilon \to 0} 0. \tag{4.8}$$

Proof. Let us first prove that there is a $\epsilon_K > 0$ such that $K \subset \rho(A_\epsilon)$ for all $\epsilon \in [0, \epsilon_K]$. Suppose that this is not the case then, there are sequences $\epsilon_n \to 0$, $\{\lambda_n\} \in K$ such that λ_n is an eigenvalue of A_{ϵ_n}. Since K is compact we may assume that there is a $\bar\lambda \in K$ such that $\lambda_n \to \bar\lambda$. It follows form Theorem 4.10, part (ii), that $\bar\lambda \in \sigma(A_0)$ which is a contradiction.

To prove (4.7), it is enough to prove that

$$\sup_{\lambda \in K,\, \epsilon \in [0,\epsilon_K]} \left\| \left(I - \lambda A_\epsilon^{-1}\right)^{-1} \right\| < \infty.$$

We assume that this is not the case; that is, assume that there are sequences $\epsilon_n \to 0$, $\lambda_n \in K$ (which we may assume convergent to $\bar\lambda \in K$) such that

$$\left\| \left(I - \lambda_n A_{\epsilon_n}^{-1}\right)^{-1} \right\| \to \infty.$$

Since $\lambda_n A_{\epsilon_n}^{-1}$ converges compactly to $\bar\lambda A_0^{-1}$ this is in contradiction with Lemma 4.7.

It remains to prove (4.8). Once again, we prove it by contradiction. Assume that there are sequences $\epsilon_n \to 0$, $K \ni \lambda_n \to \bar\lambda \in K$ and $\eta > 0$ such that

$$\left\| (\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - E_{\epsilon_n} (\lambda_n - A_0)^{-1} u \right\|_{U_{\epsilon_n}} \geqslant \eta. \tag{4.9}$$

Using the resolvent identity we have

$$(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar\lambda - A_{\epsilon_n})^{-1} E_{\epsilon_n} u = (\bar\lambda - \lambda_n)(\lambda_n - A_{\epsilon_n})^{-1} (\bar\lambda - A_{\epsilon_n})^{-1} E_{\epsilon_n} u.$$

It follows from the (4.7) that

$$\left\| (\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar\lambda - A_{\epsilon_n})^{-1} E_{\epsilon_n} u \right\|_{U_{\epsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{4.10}$$

Since, from Lemma 4.8,

$$\left\|(\bar{\lambda}-A_{\epsilon_n})^{-1}E_{\epsilon_n}u-E_{\epsilon_n}(\bar{\lambda}-A_0)^{-1}u\right\|_{U_{\epsilon_n}}\to 0\quad\text{as } n\to\infty \tag{4.11}$$

and, from the continuity properties of the resolvent operators,

$$\left\|(\lambda_n-A_0)^{-1}u-(\bar{\lambda}-A_0)^{-1}u\right\|_{U_0}\to 0\quad\text{as } n\to\infty. \tag{4.12}$$

Now, (4.10)–(4.12) are in contradiction with (4.9) and the result is proved. □

4.1. Linearization

In many instances we will be interested in analyzing the behavior, in terms of compact convergence, spectrum, etc., of operators that come from the linearization around certain stationary solutions of nonlinear problems. This amounts to study the behavior of operators of the form $A_\epsilon+V_\epsilon$ where $V_\epsilon:U_\epsilon\to U_\epsilon$ is a bounded operator (typically a multiplication by a potential). We will see that under fairly general hypotheses, once compact convergence of A_ϵ^{-1} to A_0^{-1} is obtained, we can analyze the operators of the form $A_\epsilon+V_\epsilon$.

Consider the following hypothesis

$$\boxed{\text{(4.5) holds and } V_\epsilon\in\mathcal{L}(U_\epsilon,U_\epsilon),\ \epsilon\in[0,1] \text{ such that } A_\epsilon^{-1}V_\epsilon\xrightarrow{CC}A_0^{-1}V_0.} \tag{4.13}$$

Example 4.12. Assume we are in the setting of Examples 4.1 and 4.6 and let $V_\epsilon\in L^\infty(\Omega_\epsilon)$ and $V_0\in L^\infty(\Omega)\oplus L^\infty(0,1)$ be potentials satisfying that $V_\epsilon\xrightarrow{E}V_0$. Then, we have

$$A_\epsilon^{-1}V_\epsilon\xrightarrow{CC}A_0^{-1}V_0.$$

Note that $A_\epsilon^{-1}V_\epsilon(u_\epsilon)=A_\epsilon^{-1}(V_\epsilon u_\epsilon)$. To prove this, notice that by the boundedness of the potentials V_ϵ it is easy to see that if u_ϵ is a bounded sequence in U_ϵ^p, then $V_\epsilon u_\epsilon$ is also a bounded sequence in U_ϵ^p. By the compact convergence of A_ϵ^{-1} we get that $A_\epsilon^{-1}(V_\epsilon u_\epsilon)$ is precompact.

Moreover, if $u_\epsilon\xrightarrow{E}u_0$ in U_ϵ^p, then $V_\epsilon u_\epsilon\xrightarrow{E}V_0u_0$. And therefore $A_\epsilon^{-1}V_\epsilon u_\epsilon\xrightarrow{E}A_0^{-1}V_0u_0$ since $A_\epsilon^{-1}\xrightarrow{EE}A_0^{-1}$.

We assume the following condition

$$\boxed{0\notin\sigma(A_0+V_0).} \tag{4.14}$$

It is clear that A_0+V_0 has compact resolvent. Let $\bar{A}_\epsilon=A_\epsilon+V_\epsilon$, $0\leqslant\epsilon\leqslant 1$. We can show the following result:

Proposition 4.13. *Assume that conditions* (4.13) *and* (4.14) *are satisfied. Then, there is an* $\epsilon_0>0$ *such that* $0\notin\sigma(A_\epsilon+V_\epsilon)$, $\|(A_\epsilon+V_\epsilon)^{-1}\|_{\mathcal{L}(U_\epsilon)}\leqslant M$ *independent of* ϵ *for* $0\leqslant\epsilon\leqslant\epsilon_0$. *Moreover,*

$$(A_\epsilon+V_\epsilon)^{-1}\xrightarrow{CC}(A_0+V_0)^{-1}.$$

In particular, the operators $\bar{A}_\epsilon=A_\epsilon+V_\epsilon$, $0\leqslant\epsilon\leqslant 1$, *satisfy condition* (4.5).

Proof. To prove the result note that

$$(A_\epsilon + V_\epsilon)^{-1} = \big(I + A_\epsilon^{-1} V_\epsilon\big)^{-1} A_\epsilon^{-1}.$$

Since $-A_\epsilon^{-1} V_\epsilon$ converges compactly to $-A_0^{-1} V_0$ and $-A_\epsilon^{-1}$ converges compactly to $(-A_0)^{-1}$, the uniform boundedness follows from Lemma 4.7.

To prove that $(A_\epsilon + V_\epsilon)^{-1} \xrightarrow{CC} (A_0 + V_0)^{-1}$ we note that, for each sequence $u_\epsilon \in U_\epsilon$ with $\|u_\epsilon\|_{U_\epsilon} \leqslant 1$ we have

$$v_\epsilon = (A_\epsilon + V_\epsilon)^{-1} u_\epsilon = \big(I + A_\epsilon^{-1} V_\epsilon\big)^{-1} A_\epsilon^{-1} u_\epsilon$$

is a bounded sequence and that

$$v_\epsilon = -A_\epsilon^{-1} V_\epsilon v_\epsilon + A_\epsilon^{-1} u_\epsilon.$$

Taking subsequences we may assume that $\{A_\epsilon^{-1} V_\epsilon v_\epsilon\}$ and $\{A_\epsilon^{-1} u_\epsilon\}$ are convergent and it follows that $\{v_\epsilon\}$ has a convergent subsequence. In addition, if $\{u_\epsilon\}$ is convergent to u we have that from the above that $\{v_\epsilon\}$ converges along subsequences to v which must satisfy

$$v = -A_0^{-1} V_0 v + A_0^{-1} u$$

and $v = (A_0 + V_0)^{-1} u$. From the fact that the limit is independent of the subsequence we have convergence. □

Observing that, from Proposition 4.13, $\bar{A}_\epsilon^{-1}$ converges compactly to $\bar{A}_0^{-1}$ and proceeding exactly as in Proposition 4.11 we obtain the following result:

Corollary 4.14. *Under the conditions of Proposition* 4.13*, all the results of Theorem* 4.10 *and Proposition* 4.11*, apply to the family of operators* $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$, $0 \leqslant \epsilon \leqslant 1$.

Proof. Just observe that from Proposition 4.13 the operators $\bar{A}_\epsilon$ satisfy condition (4.5). □

5. Continuity of the set of equilibria

Let us consider in the family of Banach spaces U_ϵ the following family of nonlinear problems

$$A_\epsilon u_\epsilon + f_\epsilon(u_\epsilon) = 0, \tag{5.1}$$

where $f_\epsilon : U_\epsilon \to U_\epsilon$ is a bounded and differentiable map for $\epsilon \in [0,1]$. Let $\mathcal{E}_\epsilon = \{u_\epsilon^*\colon\ A_\epsilon u_\epsilon^* + f_\epsilon(u_\epsilon^*) = 0\}$, $\epsilon \in [0,1]$.

We assume that

$$\boxed{A_\epsilon \text{ satisfies (4.5) and } A_\epsilon^{-1} f_\epsilon(\cdot) \xrightarrow{CC} A_0^{-1} f_0(\cdot).} \tag{5.2}$$

Example 5.1. Let Ω_ϵ be the dumbbell domain defined in Section 2 and consider the setting from Examples 4.1 and 4.6. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function with bounded derivatives up to second order. Let us show that if we denote by $f_\epsilon : U_\epsilon^p \to U_\epsilon^p$, $p > N$, the Nemitskĭi map of f in U_ϵ^p; that is, $f_\epsilon(u_\epsilon)(x) = f(u_\epsilon(x))$, $x \in \Omega_\epsilon$, then (5.2) is satisfied.

Suppose that $U_\epsilon^p \ni u_\epsilon \xrightarrow{E} u \in U_0^p$. Then,

$$\left\| f_\epsilon^e(u_\epsilon) - E_\epsilon f_0^e(u)\right\|_{U_\epsilon^p} = \left\| f_\epsilon^e(u_\epsilon) - f_\epsilon^e(E_\epsilon u)\right\|_{U_\epsilon^p} \leqslant L\|u_\epsilon - E_\epsilon u\|_{U_\epsilon^p} \xrightarrow{\epsilon\to 0} 0. \tag{5.3}$$

Condition (5.2) now follows from the compact convergence $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$, from the fact that f_ϵ^e is bounded uniformly for $\epsilon \in [0,1]$ and from (5.3).

Consider the following definition of the index. We refer to [33,39] for details.

Definition 5.2. Let U be a real Banach space, $\mathcal{O} \subset U$ and denote by $\mathcal{K}(\mathcal{O})$ the set of compact maps from $\mathcal{O}$ into U. We say that a triple $(I - F, \mathcal{O}, u)$ is admissible if $\mathcal{O} \subset U$ is open and bounded, $F \in \mathcal{K}(\mathcal{O})$ and $u \notin (I-F)(\partial\mathcal{O})$. A function γ which assigns an integer number $\gamma(I-F,\mathcal{O},u)$ to each admissible triple $(I-F,\mathcal{O},u)$ with the properties

(1) $\gamma(I,\mathcal{O},u) = 1$ for $u \in \mathcal{O}$;
(2) $\gamma(I-F,\mathcal{O},u) = \gamma(I-F,\mathcal{O}_1,u) + \gamma(I-F,\mathcal{O}_2,u)$ whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open subsets of $\mathcal{O}$ such that $u \notin (I-F)(\overline{\mathcal{O}} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2))$;
(3) $\gamma(I - H(t,\cdot),\mathcal{O},u(t))$ is independent of $t \in [0,1]$ whenever $H:[0,1]\times\overline{\mathcal{O}} \to U$ is compact, $u(\cdot):[0,1]\to U$ is continuous and $u(t) \notin (I - H(t,\cdot))(\partial\mathcal{O})$ on $[0,1]$

is called a Leray–Schauder degree.

Let $F \in \mathcal{K}(\mathcal{O})$, $u \in \mathcal{O}$ and $\epsilon_0 > 0$. If, for all $\epsilon \in (0,\epsilon_0]$, $(I-F, B(u,\epsilon), u)$ is an admissible triple ($B(u,\epsilon)$ is the ball of radius ϵ around u) and $\gamma(I-F,B(u,\epsilon),u)$ is independent of $\epsilon \in (0,\epsilon_0]$, we say that this common value is the index of u relatively to the map $I-F$ and denote it by $\mathrm{ind}(u, I-F)$.

Now we can show:

Theorem 5.3. *If u_0^* is an equilibrium point of* (5.1) *with $\epsilon = 0$ which satisfies $0 \notin \sigma(A_0 + f_0'(u_0^*))$ then, u_0^* is an isolated equilibrium point with $|\mathrm{ind}(u_0^*, I + A_0^{-1} f_0(u_0^*))| = 1$.*

Proof. Note that u_0^* is a solution of (5.1) with $\epsilon = 0$ if and only if it is a fixed point of the compact operator $-A_0^{-1} f_0(\cdot) : U_0 \to U_0$. Also, $0 \notin \sigma(A_0 + f_0'(u_0^*))$ if and only if $1 \notin \sigma(-A_0^{-1} f_0'(u_0^*))$. It follows that there is a constant $\eta > 0$ such that $\|v + A_0^{-1} f_0'(u_0^*)v\|_{U_0} \geqslant 2\eta\|v\|_{U_0}$. If we define $w_0(u_0^*, v) = A_0^{-1} f_0(u_0^* + v) - A_0^{-1} f_0(u_0^*) - A_0^{-1} f_0'(u_0^*)v$, then, by the differentiability of f_0 we have that

$$\frac{\|w_0(u_0^*, v)\|_{U_0}}{\|v\|_{U_0}} \xrightarrow{v\to 0} 0.$$

In particular, there is $r > 0$ such that $\|w_0(u_0^*, v)\|_{U_0} \leqslant \eta\|v\|_{U_0}$ for $\|v\|_{U_0} \leqslant r$. Then, for $\|u_0^* - u\|_{U_0} \leqslant r$ we have

$$\begin{aligned}\|u + A_0^{-1} f_0(u)\|_{U_0} &= \|u - u_0^* - (A_0^{-1} f_0(u) - A_0^{-1} f_0(u_0^*))\|_{U_0} \\ &\geqslant \|u - u_0^* + A_0^{-1} f_0'(u_0^*)(u - u_0^*)\|_{U_0} - \|w_0(u_0^*, u - u_0^*)\|_{U_0} \geqslant \eta\|u - u_0^*\|.\end{aligned}$$

Thus u_0^* is an isolated equilibrium. The proof that $|\mathrm{ind}(u_0^*, I + A_0^{-1} f_0'(u_0^*))| = 1$ follows as a direct consequence of [33, Theorem 21.6]. □

Corollary 5.4. *If u_0^* is a hyperbolic solution of* (5.1) *with $\epsilon = 0$ then, u_0^* is an isolated equilibrium and $|\mathrm{ind}(u_0^*, I + A_0^{-1} f_0'(u_0^*))| = 1$.*

Proposition 5.5. *If all points in $\mathcal{E}_0$ are isolated then, there is only a finite number of them. If $0 \notin \sigma(A_0 + f_0'(u_0^*))$ for each $u_0^* \in \mathcal{E}_0$ then, $\mathcal{E}_0$ is a finite set with an odd number of elements.*

Proof. First we observe that all solutions of (5.1) with $\epsilon = 0$ satisfies

$$u + A_0^{-1} f_0(u) = 0. \tag{5.4}$$

If we consider the ball of radius larger than $\|A_0^{-1}\|K$, with $K = \sup\{\|f_0(u)\|_{U_0}\colon u \in U_0\}$, then the operator $-A_0^{-1} f(\cdot)$ maps the ball $B(0, \|A_0^{-1}\|K) \subset U_0$ into itself. By Schauder fixed point theorem [33, Theorem 21.6] $\gamma(I + A_0^{-1} f_0(\cdot), B(0, \|A_0^{-1}\|K), 0) = 1$ (γ is the degree of Leray–Schauder) and there is at least one fixed point u_0^* for $-A_0^{-1} f(\cdot)$ in $B(0, \|A_0^{-1}\|K)$; that is,

$$u_0^* + A_0^{-1} f_0(u_0^*) = 0 \quad \text{with } u_0^* \in B(0, \|A_0^{-1}\|K).$$

Since the operator $-A_0^{-1} f_0(\cdot)\colon U_0 \to U_0$ is compact we have that the set $\mathcal{E}_0 = \{u\colon A_0 u + f_0(u) = 0\}$ is compact in U_0. Moreover, by Theorem 5.3 any fixed point u_0^* is isolated. If the number of the fixed points is infinite, i.e., we have a sequence $\{u_i^*\}_{i=1}^{\infty}$, then the sequence $-A_0^{-1} f_0(u_i^*) = u_i^* \to u_\infty^*$ converges on some subsequence $i \in \mathbb{N}' \subset \mathbb{N}$, which is a contradiction with the fact that each fixed point u_∞^* is isolated. So the number of the equilibrium points is finite. Now by [33, Theorem 20.6]

$$1 = \gamma(I + A_0^{-1} f_0(\cdot), B(0, \|A_0^{-1}\|K), 0) = \sum_{i=1}^{d} \mathrm{ind}(u_i^*, I + A_0^{-1} f(\cdot))$$

and therefore the number $d = 2k + 1$ for some integer $k \geqslant 0$. □

Proposition 5.6. *Assume that condition* (5.2) *is satisfied and that problems* (5.1) *have solutions $\{u_\epsilon^*\}$, $\epsilon \in (0, 1]$. Then, taking subsequences if necessary, there is a solution u_0^* of* (5.1) *with $\epsilon = 0$ such that $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} \to 0$ as $\epsilon \to 0$.*

Proof. If u_ϵ^* is a solution of (5.1) we have that $u_\epsilon^* = -A_\epsilon^{-1} f_\epsilon(u_\epsilon^*)$. From the fact that $A_\epsilon^{-1} f_\epsilon(\cdot)\colon U_\epsilon \to U_\epsilon$ is bounded uniformly for $\epsilon \in [0, 1]$ it follows that $\{u_\epsilon^*\}$ is bounded. From (5.2), we have that there is a subsequence, which we again denote by u_ϵ^*, such that $u_\epsilon^* \xrightarrow{E} u_0^*$. Again from (5.2) we have that

$$u_0^* + A_0^{-1} f_0(u_0^*) = 0,$$

which is equivalent to say that u_0^* is a solution of (5.1) with $\epsilon = 0$. □

Proposition 5.7. *Assume that* (5.2) *holds and that u_0^* is hyperbolic solution of* (5.1) *with $\epsilon = 0$. Then there are ϵ_0 and $\delta > 0$ such that for $0 < \epsilon \leqslant \epsilon_0$ Eqs.* (5.1) *have at least one solution u_ϵ^* in $\{w_\epsilon \colon \|w_\epsilon - E_\epsilon u_0^*\|_{U_\epsilon} \leqslant \delta\}$. Furthermore, $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} \to 0$ as $\epsilon \to 0$.*

Proof. As in Corollary 5.4 there is a ball $B(u_0^*, \delta)$ such that there are no other fixed points in it except u_0^* and we get $|\mathrm{ind}(u_0^*, I + A_0^{-1} f_0(\cdot))| = 1$. It is easy to see that the hypotheses of [39, Theorem 3] are satisfied and therefore there is at least one fixed point u_ϵ^* in any ball $B(E_\epsilon u_0^*, \delta)$, $\epsilon \leqslant \epsilon_0$, for some $\epsilon_0 > 0$. This sequence $\{u_\epsilon^*\}$ is E-convergent to u_0^*. □

The last two results, Propositions 5.6 and 5.7, show the continuity of the set of equilibria in the following sense: if u_ϵ^* is a sequence of equilibria of (5.1) then we can get a subsequence such that $u_\epsilon^* \xrightarrow{E} u_0^*$, which is an equilibrium of the limit equations and vice versa, if u_0^* is an equilibrium solution of the limit equation which is hyperbolic, then there exists a sequence of solutions u_ϵ^* for all $\epsilon > 0$ small enough such that $u_\epsilon^* \xrightarrow{E} u_0^*$.

We want to impose conditions now on the nonlinearities f_ϵ that guarantee that for a fixed hyperbolic equilibrium solution u_0^* of the limit equation we have one and only one solution u_ϵ^* of the perturbed equation nearby. In order to accomplish this, we will need some kind of uniform differentiability property of the nonlinearities f_ϵ. For this, define first

$$w_\epsilon\big(u_\epsilon^*, v\big) = A_\epsilon^{-1} f_\epsilon\big(u_\epsilon^* + v\big) - A_\epsilon^{-1} f_\epsilon\big(u_\epsilon^*\big) - A_\epsilon^{-1} f_\epsilon'\big(u_\epsilon^*\big)v.$$

Consider the following hypothesis

$$\boxed{\begin{array}{l}\text{Hypothesis (5.2) holds, and if } u_\epsilon^* \text{ are equilibrium solutions with } u_\epsilon^* \xrightarrow{E} u_0^* \text{ then,}\\ A_\epsilon^{-1} f_\epsilon'(u_\epsilon^*) \xrightarrow{CC} A_0^{-1} f_0'(u_0^*) \text{ and } \frac{\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon}}{\|v\|_{U_\epsilon}} = o(1) \text{ as } \|v\|_{U_\epsilon} \to 0, \text{ uniformly in } \epsilon.\end{array}} \tag{5.5}$$

Observe that saying that $\frac{\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon}}{\|v\|_{U_\epsilon}} = o(1)$ uniformly in ϵ means that for each $\mu > 0$, there exists a $\delta > 0$ such that $\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon} \leqslant \mu \|v\|_{U_\epsilon}$ for all $v \in U_\epsilon$ with $\|v\|_{U_\epsilon} \leqslant \delta$.

We can show now the following theorem.

Theorem 5.8. *Assume* (5.5) *holds and let u_0^* be a solution of* (5.1) *with $\epsilon = 0$ which satisfies $0 \notin \sigma(A_0 + f_0'(u_0^*))$. Then, there is a $\delta > 0$ such that* (5.1) *has a unique solution u_ϵ^* such that $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} < \delta$.*

If, for all solutions u_0^* of (5.1) *with $\epsilon = 0$, $0 \notin \sigma(A_0 + f_0'(u_0^*))$ then, from Proposition* 5.5, (5.1) *with $\epsilon = 0$ has a finite number n_0 of solutions $u_1^*, \ldots, u_{n_0}^*$. In this case, there is an ϵ_0 such that* (5.1) *has exactly n_0 solutions, $u_{\epsilon,1}^*, \ldots, u_{\epsilon,n_0}^*$, for all $\epsilon \leqslant \epsilon_0$ and $u_{\epsilon,i}^* \xrightarrow{E} u_i^*$, $1 \leqslant i \leqslant n_0$.*

If, moreover, u_0^* is a hyperbolic equilibrium point then u_ϵ^* is also hyperbolic and we can apply Corollary 4.14. *In particular, the linear unstable manifold of u_ϵ^* E-converges to the linear unstable manifold of u_0^*.*

Proof. Note that u_ϵ^* is a solution of (5.1) if and only if it is a fixed point of the compact operator $-A_\epsilon^{-1} f_\epsilon(\cdot) : U_\epsilon \to U_\epsilon$. Also, from Lemma 4.7, there is an $\epsilon_0 > 0$ and $\eta > 0$ (independent of $\epsilon \in [0, \epsilon_0]$) such that, for any $\epsilon \leqslant \epsilon_0$, $0 \notin \sigma(A_\epsilon + f_\epsilon'(u_\epsilon^*))$ and $\|(I + A_\epsilon^{-1} f_\epsilon'(u_\epsilon^*))v_\epsilon\|_{U_\epsilon} \geqslant$

$2\eta\|v_\epsilon\|_{U_\epsilon}$. We write

$$A_\epsilon^{-1} f_\epsilon\big(u_\epsilon^* + v_\epsilon\big) - A_\epsilon^{-1} f_\epsilon\big(u_\epsilon^*\big) - A_\epsilon^{-1} f_\epsilon'\big(u_\epsilon^*\big)v_\epsilon = w_\epsilon\big(u_\epsilon^*, v_\epsilon\big),$$
$$\frac{\|w(u_\epsilon^*, v_\epsilon)\|_{U_\epsilon}}{\|v_\epsilon\|_{U_\epsilon}} \leqslant h\big(\|v_\epsilon\|_{U_\epsilon}\big),$$

where (from (5.5)) $h:[0,\infty) \to \mathbb{R}$ can be taken continuous with $h(0) = 0$. Hence, there is a $\delta > 0$ (independent of ϵ) such that $\|w_\epsilon(u_\epsilon^*, v_\epsilon)\|_{U_\epsilon} \leqslant \eta\|v_\epsilon\|_{U_\epsilon}$ for $\|v_\epsilon\|_{U_0} \leqslant 2\delta$. Then, for $\|u_\epsilon^* - u_\epsilon\|_{U_\epsilon} \leqslant 2\delta$

$$\begin{aligned}\big\|u_\epsilon + A_\epsilon^{-1} f_\epsilon(u_\epsilon)\big\|_{U_\epsilon} &\geqslant \big\|u_\epsilon - u_\epsilon^* + A_\epsilon^{-1} f_\epsilon'\big(u_\epsilon^*\big)\big(u_\epsilon^* - u_\epsilon\big)\big\|_{U_\epsilon} - \big\|w_\epsilon\big(u_\epsilon^*, u_\epsilon - u_\epsilon^*\big)\big\|_{U_\epsilon}\\ &\geqslant \eta\big\|u_\epsilon - u_\epsilon^*\big\|.\end{aligned}$$

Thus u_ϵ^* is the only solution of (5.1) in $B_{2\delta}(u_\epsilon^*)$. This together with the fact that $u_\epsilon \xrightarrow{E} u_0^*$ implies the result. □

Example 5.9. Assume we are exactly in the same conditions of Example 5.1. Let us show that hypotheses (5.5) also holds. Notice that if $u_\epsilon^* \xrightarrow{E} u_0^*$, and if we define $V_\epsilon = f'(u_\epsilon^*)$, $V_0 = f'(u_0)$, we have that since f' is a bounded function that $V_\epsilon \in L^\infty(\Omega_\epsilon)$, $V_0 \in L^\infty(\Omega) \oplus L^\infty(0,1)$. Moreover, $V_\epsilon \xrightarrow{E} V_0$. Applying the results in Example 4.12, we get

$$A_\epsilon^{-1} f'\big(u_\epsilon^*\big) \xrightarrow{CC} A_0^{-1} f'\big(u_0^*\big).$$

Let us prove now that, for each $\epsilon \in [0,1]$, we get

$$\big\|w_\epsilon(u_\epsilon, v)\big\|_{U_\epsilon^p} := \big\|A_\epsilon^{-1}\big(f_\epsilon^e(u_\epsilon + v) - f_\epsilon^e(u_\epsilon) - \big(f^e\big)'(u_\epsilon)v\big)\big\|_{U_\epsilon^p} \leqslant C\|v\|_{U_\epsilon^p}^{\frac{p}{q}}, \quad v \in U_\epsilon^p, \tag{5.6}$$

for any $N < q < p$, where C is a constant independent of ϵ. To prove (5.6) we note first that, as it will be proved in Appendix A, Lemma A.11, we have that for each $N < q$ there exists a constant C, independent of ϵ, such that

$$\big\|A_\epsilon^{-1}\big\|_{\mathcal{L}(U_\epsilon^q, L^\infty(\Omega_\epsilon))} \leqslant C. \tag{5.7}$$

By interpolation, it is not difficult to see that if $N < q < p$ we also have $\|A_\epsilon^{-1}\|_{\mathcal{L}(U_\epsilon^q, U_\epsilon^p)} \leqslant C$.
Hence, if $N < q < p$, we have

$$\begin{aligned}&\big\|A_\epsilon^{-1}\big(f_\epsilon^e(u_\epsilon + v) - f_\epsilon^e(u_\epsilon) - \big(f^e\big)'(u_\epsilon)v\big)\big\|_{U_\epsilon^p}\\ &\quad\leqslant C\big\|f(u_\epsilon + v) - f(u_\epsilon) - f'(u_\epsilon)v\big\|_{U_\epsilon^q}\\ &\quad\leqslant C\big\|\big[f'\big(u_\epsilon(x) + \theta(x)v(x)\big) - f'\big(u_\epsilon(x)\big)\big]v(x)\big\|_{U_\epsilon^q}\\ &\quad\leqslant C\big\|f'\big(u_\epsilon(x) + \theta(x)v(x)\big) - f'\big(u_\epsilon(x)\big)\big\|_{U_\epsilon^r}\big\|v(x)\big\|_{U_\epsilon^p},\end{aligned} \tag{5.8}$$

where $\frac{1}{r}+\frac{1}{p}=\frac{1}{q}$. Note that

$$\begin{aligned}\big\|f'\big(u_\epsilon(x)+\theta(x)v(x)\big)-f'\big(u_\epsilon(x)\big)\big\|_{L^\infty(\Omega)}&\leqslant C,\\ \big\|f'\big(u_\epsilon(x)+\theta(x)v(x)\big)-f'\big(u_\epsilon(x)\big)\big\|_{U_\epsilon^p}&\leqslant C\|v\|_{U_\epsilon^p}\end{aligned}$$

and by interpolation

$$\big\|f'\big(u_\epsilon(x)+\theta(x)v(x)\big)-f'\big(u_\epsilon(x)\big)\big\|_{U_\epsilon^r}\leqslant C\|v\|_{U_\epsilon^p}^{\frac{p}{r}}\leqslant C\|v\|_{U_\epsilon^p}^{\frac{p-q}{q}}$$

which implies (5.6).

6. Proof of the main results: Theorems 2.3 and 2.5

In this section we will assume that Proposition 2.7 is proved and will provide a demonstration of Theorems 2.3 and 2.5. The proof of Proposition 2.7 will be obtained in Appendix A.

Proof of Theorem 2.3. Under the conditions of the nonlinearity from Section 2 and with the aid of the maximum principle, we easily get that the set of equilibrium points $\mathcal{E}_\epsilon$ is bounded in $L^\infty(\Omega_\epsilon)$ with a bound independent of ϵ. Similarly, the set of equilibria of the limit problem is also uniformly bounded.

Notice that, if $p>N$, with the definitions of U_ϵ^p and U_0^p from Section 2 and Example 4.1, we have from Proposition 2.7 the compact convergence of A_ϵ^{-1} to A_0^{-1}. In particular, (4.5) holds true. Moreover, as it is shown in Example 5.1, condition (5.2) is also satisfied. Applying now Proposition 5.6, we show (2.7).

If we denote now by $f_\epsilon^*=f(u_\epsilon^*)\in U_\epsilon^p$ and $f_0^*=f(u_0^*)\in U_0^p$, by (2.7) and by the continuity of the nonlinearity f, we have that $\|f_\epsilon^*-E_\epsilon f_0^*\|_{U_\epsilon^p}\to 0$ as $\epsilon\to 0$. Applying Proposition 2.7, point (2)(i)–(iii) and taking into account that $u_\epsilon^*=A_\epsilon^{-1}f_\epsilon^*$, $u_0^*=A_0^{-1}f_0^*$, we prove (i) of Theorem 2.3.

To show (ii), observe that by Example 5.9, we have that hypothesis (5.5) holds true. In particular, we can apply Theorem 5.8, which proves (ii). This concludes the proof of the theorem. □

Proof of Theorem 2.5. If we are in the conditions of Theorem 2.3 and we have a sequence of equilibria u_ϵ^* which E-converges to $u_0^*=(w_0,v_0)$ satisfying (2.7) and (2.8), we have that if we define $V_\epsilon=f'(u_\epsilon^*)+M$ and $V_0=f'(u_0^*)+M$, for some positive M large enough so that $f'(u_0^*)+M\geqslant 0$, then, as it is shown in Example 4.12, (4.13) holds. Moreover, since $f'(u_0^*)+M\geqslant 0$, we have that (4.14) also holds.

Hence, we can apply Proposition 4.13 which in particular implies that the spectral convergence result given by Theorem 4.10 hold true for the operators $A_\epsilon+f'(u_\epsilon^*)+M$ and $A_0+f'(u_0^*)+M$. Since the effect of the constant M in the operators above is just a shift in the spectrum, we show that the results of Theorem 4.10 hold true for the operators $A_\epsilon+f'(u_\epsilon^*)$ and $A_0+f'(u_0^*)$. In particular, we obtain the convergence of the eigenvalues and the convergence of the spectral projections in U_ϵ^p. To show the convergence in the H_ϵ^1 norm we proceed similarly as in Theorem 2.3. □

Appendix A. Resolvent convergence

In this appendix we will show Proposition 2.7, which is the main result on the convergence of the resolvent operators.

Before we start comparing the resolvent operators of A_ϵ and A_0, we present some preliminary results, including some extension and projection operators, that will be needed to prove the result.

A.1. The projection

We introduce now the basic projection operator that we will use.

Let $\psi_\epsilon \in U_\epsilon^p$ where $U_\epsilon^p = L^p(\Omega_\epsilon)$ with the norm

$$\|\phi_\epsilon\|_{U_\epsilon} = \|\phi_\epsilon\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|\phi_\epsilon\|_{L^p(R_\epsilon)},$$

for $\epsilon > 0$ and $U_0^p = L^p(\Omega) \oplus L_g^p(0,1)$ with the norm

$$\big\|(w,v)\big\|_{U_0^p} = \|w\|_{L^p(\Omega)} + \|v\|_{L_g^p(0,1)},$$

where $\|w\|_{L_g^p(0,1)} = (\int_0^1 |w(s)|^p g(s)\,ds)^{1/p}$.

To compare functions from U_ϵ^p and from U_0^p, we define the following projection operator

$$M_\epsilon\colon \quad U_\epsilon^p \to U_0^p, \quad \psi_\epsilon \to (M_\epsilon \psi_\epsilon)(x) = \begin{cases} \psi_\epsilon(x), & x \in \Omega, \\ T_\epsilon^s \psi_\epsilon, & s \in (0,1), \end{cases} \tag{A.1}$$

where

$$T_\epsilon^s \psi_\epsilon(x) = \frac{1}{|\Gamma_\epsilon^s|} \int\limits_{\Gamma_\epsilon^s} \psi(s,y)\,dy, \quad \Gamma_\epsilon^s = \big\{y\colon (s,y) \in R_\epsilon\big\}. \tag{A.2}$$

The following result holds:

Lemma A.1. *The projection M_ϵ is a bounded operator with norm $\|M_\epsilon\|_{\mathcal{L}(U_\epsilon, U_0^p)} = 1$.*

Proof. If $\phi_\epsilon \in U_\epsilon$ then, if $x = (s,y)$ with $s \in \mathbb{R}$ and $y \in \mathbb{R}^{N-1}$,

$$\begin{aligned}
\|M_\epsilon \phi_\epsilon\|_{U_0^p} &= \left(\int\limits_\Omega \big|\phi_\epsilon(x)\big|^p\,dx\right)^{\frac{1}{p}} + \left(\int\limits_0^1 g(s)\big|M_\epsilon\phi_\epsilon(s)\big|^p\,ds\right)^{\frac{1}{p}} \\
&= \left(\int\limits_\Omega \big|\phi_\epsilon(x)\big|^p\,dx\right)^{\frac{1}{p}} + \left(\int\limits_0^1 g(s)\left|\frac{1}{|\Gamma_\epsilon^s|}\int\limits_{\Gamma_\epsilon^s} \phi_\epsilon(s,y)\,dy\right|^p ds\right)^{\frac{1}{p}} \\
&= \left(\int\limits_\Omega \big|\phi_\epsilon(x)\big|^p\,dx\right)^{\frac{1}{p}} + \epsilon^{1-N}\left(\int\limits_0^1 g(s)^{-p+1}\left|\int\limits_{\Gamma_\epsilon^s} \phi_\epsilon(s,y)\,dy\right|^p ds\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leqslant \left(\int_{\Omega} \left|\phi_\epsilon(x)\right|^p dx\right)^{\frac{1}{p}} + \epsilon^{1-N}\left(\int_0^1 g(s)^{-p+1}\left|\Gamma_\epsilon^s\right|^{p-1}\int_{\Gamma_\epsilon^s}\left|\phi_\epsilon(s,y)\right|^p dy\,ds\right)^{\frac{1}{p}}$$

$$= \left(\int_{\Omega} \left|\phi_\epsilon(x)\right|^p dx\right)^{\frac{1}{p}} + \epsilon^{\frac{1-N}{p}}\left(\int_0^1 \int_{\Gamma_\epsilon^s}\left|\phi_\epsilon(s,y)\right|^p dy\,ds\right)^{\frac{1}{p}} = \|\phi_\epsilon\|_{U_\epsilon}.$$

The equality holds if ϕ_ϵ is independent of y in R_ϵ. □

A.2. The extension

Let $\psi \in U_0^p$, to consider ψ as a function in U_ϵ^p, we define the following extension operator

$$E_\epsilon: \quad U_0^p \to U_\epsilon, \quad \psi \to (E_\epsilon \psi)(x) = \begin{cases} \psi(x), & \Omega, \\ \psi(s), & (s,y) \in R_\epsilon. \end{cases} \tag{A.3}$$

Of course, E_ϵ can be considered in larger spaces with the same definition. It is easy to see that E_ϵ has the following property:

Lemma A.2. *$E_\epsilon : U_0^p \to U_\epsilon^p$ is a bounded linear operator and*

$$\left\|E_\epsilon(w,v)\right\|_{U_\epsilon^p} = \left\|(w,v)\right\|_{U_0^p},$$

for all $(w,v) \in U_0^p$.

Lemma A.3. *There is a positive constant C such that, for $\psi_\epsilon \in H^1(\Omega_\epsilon)$, we have that*

$$\|\psi_\epsilon\|^2_{L^2(R_\epsilon)} = \|\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon\|^2_{L^2(R_\epsilon)} + \|E_\epsilon M_\epsilon \psi_\epsilon\|^2_{L^2(R_\epsilon)}, \tag{A.4}$$

$$\|E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon\|^2_{L^2(R_\epsilon)} \leqslant C\epsilon^2 \left\|\frac{\partial \psi_\epsilon}{\partial y}\right\|^2_{L^2(R_\epsilon)}. \tag{A.5}$$

Proof. Note that

$$\|\psi_\epsilon\|^2_{L^2(R_\epsilon)} = \int_{R_\epsilon} |\psi_\epsilon|^2\,dx = \int_{R_\epsilon} \left|(\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) + E_\epsilon M_\epsilon \psi_\epsilon\right|^2 dx$$

$$= \int_{R_\epsilon} |\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon|^2 + 2\int_{R_\epsilon} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon + \int_{R_\epsilon} |E_\epsilon M_\epsilon \psi_\epsilon|^2.$$

On the other hand,

$$\int_{R_\epsilon} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon = \int_0^1 \int_{\Gamma_\epsilon^s} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon\,ds\,dy$$

$$= \int_0^1 M_\epsilon \psi_\epsilon(x) \left\{ \int_{\Gamma_\epsilon^s} \left[\psi_\epsilon(x) - E_\epsilon M_\epsilon \psi_\epsilon(x) \right] dy \right\} ds = 0.$$

So identity (A.4) follows. Observe that

$$\|E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon\|_{L^2(R_\epsilon)}^2 = \int_{R_\epsilon} \left|(E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)\right|^2 dx = \int_0^1 \int_{\Gamma_\epsilon^s} \left|(E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)\right|^2 ds\, dy.$$

Hence, let us estimate $\int_{\Gamma_\epsilon^s} |(M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)|^2 \, dy$. In fact, from the variational characterization of eigenvalues for the Neumann Laplacian in Γ_ϵ^s, we have that

$$\lambda_2\left(\Gamma_\epsilon^s\right) = \min\left\{ \frac{\int_{\Gamma_\epsilon^s} |\nabla \phi|^2}{\int_{\Gamma_\epsilon^s} |\phi|^2} \colon \phi \in H^1\left(\Gamma_\epsilon^s\right),\ \phi \neq 0,\ \int_{\Gamma_\epsilon^s} \phi = 0 \right\}. \tag{A.6}$$

Taking $\phi = M_\epsilon \psi_\epsilon - \psi_\epsilon$, we have

$$\int_{\Gamma_\epsilon^s} |M_\epsilon \psi_\epsilon - \psi_\epsilon|^2 \leqslant \frac{1}{\lambda_2(\Gamma_\epsilon^s)} \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2. \tag{A.7}$$

From (A.6), it follows that

$$\begin{aligned} \lambda_2\left(\Gamma_\epsilon^s\right) &= \min\left\{ \frac{\int_{\Gamma_\epsilon^s} |\nabla \phi|^2}{\int_{\Gamma_\epsilon^s} |\phi|^2} \colon \phi \in H^1\left(\Gamma_\epsilon^s\right),\ \phi \neq 0,\ \int_{\Gamma_\epsilon^s} \phi = 0 \right\} \\ &= \frac{1}{\epsilon^2} \min\left\{ \frac{\int_{\Gamma_1^s} |\nabla \tilde{\phi}|^2}{\int_{\Gamma_1^s} |\tilde{\phi}|^2} \colon \tilde{\phi} \in H^1\left(\Gamma_1^s\right),\ \tilde{\phi} \neq 0,\ \int_{\Gamma_1^s} \tilde{\phi} = 0 \right\} = \frac{1}{\epsilon^2} \lambda_2\left(\Gamma_1^s\right), \end{aligned} \tag{A.8}$$

where $\lambda_2(\Gamma_1^s)$ is the second eigenvalue of the Neumann Laplacian in Γ_1^s. Using (A.8) and (A.7), we have that

$$\int_{\Gamma_\epsilon^s} |M_\epsilon \psi_\epsilon - \psi_\epsilon|^2 \leqslant \frac{\epsilon^2}{\lambda_2(\Gamma_1^s)} \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2 \leqslant C\epsilon^2 \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2, \tag{A.9}$$

where we used the fact that the map $[0,1] \ni s \to \lambda_2(\Gamma_1^s) \in (0,\infty)$ is continuous and therefore attains its minimum at a positive value; that is,

$$m := \min_{0 \leqslant s \leqslant 1} \lambda_2\left(\Gamma_1^s\right) = \lambda_2\left(\Gamma_1^{\bar{x}}\right) > 0, \quad \bar{x} \in [0,1],$$

from which we have $C := \frac{1}{m} \geqslant \frac{1}{\lambda_2(\Gamma_1^s)}$. Now, integrating from 0 to 1 we have inequality (A.5). $\square$

A.3. Continuous extension

Observe that the operator E_ϵ does not takes continuous functions into continuous functions. When such property is required we consider the following extension operator. If $\mathcal{C} = \{(w, v) \in C(\overline{\Omega}) \oplus C(0,1)$ with $w(0) = v(0)$ and $w(1) = v(1)\}$ then

$$E^\epsilon_{\mathcal{C}}:\quad \mathcal{C} \to C(\overline{\Omega}_\epsilon), \quad (w, v) \to E^\epsilon_{\mathcal{C}}(w, v) = \begin{cases} w, & x \in \Omega, \\ \tilde{v}, & x \in R_\epsilon, \end{cases} \tag{A.10}$$

where

$$\tilde{v}(x) = v(s) + h_\epsilon(s)\big(w(0, y) - v(0)\big) + h_\epsilon(1 - s)\big(w(1, y) - v(1)\big), \quad x \in R_\epsilon, \tag{A.11}$$

and $h_\delta(s) = h(\frac{s}{\delta})$, where $h : \mathbb{R}^+ \to [0, 1]$ is C^∞ function such that

$$h(s) = \begin{cases} 1, & \text{for } s \in [0, 1/4], \\ 0, & \text{for } s \geqslant 3/4 \end{cases}$$

and $|h'(s)| \leqslant C$.

We can easily estimate the difference of these operators in the following way.

Lemma A.4. *Let E_ϵ and $E^\epsilon_{\mathcal{C}}$ be the extension operators defined above. If $(w, v) \in \mathcal{C}$ with $(w, v) \in C^1(\overline{\Omega}) \oplus C^1([0, 1])$ we have that*

$$\big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|_{L^2(\Omega_\epsilon)} \leqslant C\epsilon^{\frac{N+2}{2}}\|w\|_{C^1(\overline{\Omega})}, \tag{A.12}$$

$$\big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|_{H^1(\Omega)\oplus H^1(R_\epsilon)} \leqslant C\epsilon^{N/2}\|w\|_{C^1(\overline{\Omega})}, \tag{A.13}$$

$$\big|\big\|E^\epsilon_{\mathcal{C}}(w, v)\big\|^2_{L^2(R_\epsilon)} - \big\|E_\epsilon(w, v)\big\|^2_{L^2(R_\epsilon)}\big| \leqslant C\epsilon^{N+1}\|w\|_{C^1(\overline{\Omega})}\|v\|_{C^0(0,1)}, \tag{A.14}$$

$$\big|\big\|\nabla E^\epsilon_{\mathcal{C}}(w, v)\big\|^2_{L^2(R_\epsilon)} - \big\|\nabla E_\epsilon(w, v)\big\|^2_{L^2(R_\epsilon)}\big| \leqslant C\epsilon^{N}\|w\|_{C^1(\overline{\Omega})}\|v\|_{C^1(0,1)}. \tag{A.15}$$

Proof. Let $\mathcal{C} \ni (w, v) \in C^1(\overline{\Omega}) \oplus C(0, 1)$. Then, since

$$\begin{aligned}\big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|^2_{L^2(\Omega_\epsilon)} &= \big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|^2_{L^2(\Omega)} + \big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|^2_{L^2(R_\epsilon)} \\ &= \big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|^2_{L^2(R_\epsilon)},\end{aligned}$$

we only need to estimate $\|(E_\epsilon - E^\epsilon_{\mathcal{C}})(w, v)\|^2_{L^2(R_\epsilon)}$. Hence,

$$\begin{aligned}&\big\|\big(E_\epsilon - E^\epsilon_{\mathcal{C}}\big)(w, v)\big\|^2_{L^2(R_\epsilon)} \\ &\quad= \int_0^\epsilon \int_{\Gamma^s_\epsilon} \big|h_\epsilon(s)\big(w(0, y) - v(0)\big)\big|^2\,dy\,ds + \int_{1-\epsilon}^1 \int_{\Gamma^s_\epsilon} \big|h_\epsilon(1 - s)\big(w(1, y) - v(1)\big)\big|^2\,dy\,ds\end{aligned}$$

$$= \int_0^{\epsilon} \int_{\Gamma_\epsilon^s} \left|h_\epsilon(s)\right|^2 \left|w(0,y) - w(0,0)\right|^2 dy\, ds + \int_{1-\epsilon}^{1} \int_{\Gamma_\epsilon^s} \left|h_\epsilon(1-s)\right|^2 \left|w(1,y) - w(1,0)\right|^2 dy\, ds$$

$$\leqslant C_1 \int_0^{\epsilon} \int_{\Gamma_\epsilon^s} |y|^2 \left(\sup_{y \in \Gamma_\epsilon^s} \left| \frac{\partial w}{\partial y} \right| \right)^2 dy\, ds + C_1 \int_{1-\epsilon}^{1} \int_{\Gamma_\epsilon^s} |y|^2 \left(\sup_{y \in \Gamma_\epsilon^s} \left| \frac{\partial w}{\partial y} \right| \right)^2 dy\, ds$$

$$\leqslant 2C_1 \left(\int_0^{\epsilon} \int_{\Gamma_\epsilon^s} |y|^2\, dy\, ds + \int_{1-\epsilon}^{1} \int_{\Gamma_\epsilon^s} |y|^2\, dy\, ds \right) \|w\|^2_{C^1(\overline{\Omega})} \leqslant C\epsilon^{N+2} \|w\|^2_{C^1(\overline{\Omega})}, \tag{A.16}$$

where we have used the fact that $v(0) = w(0,0)$ and $v(1) = w(1,0)$ and that $|\Gamma_\epsilon^s| \leqslant C\epsilon^{N-1}$. This shows (A.12).

To show (A.13), it is enough to estimate $\|\nabla(E_\epsilon - E_{\mathcal{C}}^\epsilon)(w,v)\|_{L^2(R_\epsilon)}$. Since $h'_\epsilon(s) = \epsilon^{-1} h'(s/\epsilon)$ and with a similar argument we have

$$\begin{aligned}
&\left\|\nabla\left(E_\epsilon - E_{\mathcal{C}}^\epsilon\right)(w,v)\right\|^2_{L^2(R_\epsilon)} \\
&\quad = \int_0^{\epsilon} \left|h'_\epsilon(s)\right|^2 \int_{\Gamma_\epsilon^s} \left|w(0,y) - v(0)\right|^2 dy\, ds + \int_0^{\epsilon} \left|h_\epsilon(s)\right|^2 \int_{\Gamma_\epsilon^s} \left|\nabla_y w(0,y)\right|^2 dy\, ds \\
&\qquad + \int_{1-\epsilon}^{1} \left|h'_\epsilon(1-s)\right|^2 \int_{\Gamma_\epsilon^s} \left|w(1,y) - v(1)\right|^2 dy\, ds + \int_{1-\epsilon}^{1} \left|h_\epsilon(1-s)\right|^2 \int_{\Gamma_\epsilon^s} \left|\nabla_y w(1,y)\right|^2 dy\, ds \\
&\quad \leqslant C_1 \epsilon^{-2} \int_0^{\epsilon} \int_{\Gamma_\epsilon^s} |y|^2\, dy\, ds\, \|w\|^2_{C^1(\overline{\Omega})} + \widetilde{C}_1 \int_0^{\epsilon} \int_{\Gamma_\epsilon^s} dy\, ds\, \|w\|^2_{C^1(\overline{\Omega})} \\
&\qquad + C_1 \epsilon^{-2} \int_{1-\epsilon}^{1} \int_{\Gamma_\epsilon^s} |y|^2\, dy\, ds\, \|w\|^2_{C^1(\overline{\Omega})} + \widetilde{C}_1 \int_{1-\epsilon}^{1} \int_{\Gamma_\epsilon^s} dy\, ds\, \|w\|^2_{C^1(\overline{\Omega})} \\
&\quad \leqslant 2C_1 \epsilon^N \|w\|^2_{C^1(\overline{\Omega})} + 2\widetilde{C}_1 \epsilon^N \|w\|^2_{C^1(\overline{\Omega})} \leqslant C\epsilon^N \|w\|^2_{C^1(\overline{\Omega})},
\end{aligned}$$

where we have also used that $\int_0^\epsilon \int_{\Gamma_\epsilon^s} dy\, ds = O(\epsilon^N)$.

The proof of the last two inequalities follows from the previous ones in the following way

$$\begin{aligned}
\left\|E_{\mathcal{C}}^\epsilon(w,v)\right\|^2_{L^2(R_\epsilon)} &= \left\|E_{\mathcal{C}}^\epsilon(w,v) - E_\epsilon(w,v) + E_\epsilon(w,v)\right\|^2_{L^2(R_\epsilon)} \\
&= \left\|E_{\mathcal{C}}^\epsilon(w,v) - E_\epsilon(w,v)\right\|^2_{L^2(R_\epsilon)} + \left\|E_\epsilon(w,v)\right\|^2_{L^2(R_\epsilon)} \\
&\quad + 2\left(E_{\mathcal{C}}^\epsilon(w,v) - E_\epsilon(w,v), E_\epsilon(w,v)\right)_{L^2(R_\epsilon)} \quad \text{and}
\end{aligned} \tag{A.17}$$

$$\begin{aligned}\left\|\nabla E_{\mathcal{C}}^{\epsilon}(w,v)\right\|_{L^2(R_\epsilon)}^2 &= \left\|\nabla E_{\mathcal{C}}^{\epsilon}(w,v)-\nabla E_\epsilon(w,v)\right\|_{L^2(R_\epsilon)}^2+\left\|\nabla E_\epsilon(w,v)\right\|_{L^2(R_\epsilon)}^2\\ &\quad+2\big(\nabla E_{\mathcal{C}}^{\epsilon}(w,v)-\nabla E_\epsilon(w,v),\nabla E_\epsilon(w,v)\big)_{L^2(R_\epsilon)}.\end{aligned}\tag{A.18}$$

But, taking into account that $E_{\mathcal{C}}^{\epsilon}(w,v)=E_\epsilon(w,v)$ apart from the set $\widetilde{R}_\epsilon=\{(s,y)\in R_\epsilon\colon\ 0<s<\epsilon, 1-\epsilon<s<1\}$ which has measure of the order of ϵ^N, then

$$\begin{aligned}&\left|\big(E_{\mathcal{C}}^{\epsilon}(w,v)-E_\epsilon(w,v),E_\epsilon(w,v)\big)_{L^2(R_\epsilon)}\right|\\ &\quad\leqslant\left\|E_{\mathcal{C}}^{\epsilon}(w,v)-E_\epsilon(w,v)\right\|_{L^2(R_\epsilon)}\left\|E_\epsilon(w,v)\right\|_{L^2(\widetilde{R}_\epsilon)}\\ &\quad\leqslant C\epsilon^{\frac{N+2}{2}}\|w\|_{C^1(\overline{\Omega})}\epsilon^{\frac{N}{2}}\|v\|_{C(0,1)}\leqslant C\epsilon^{N+1}\|w\|_{C^1(\overline{\Omega})}\|v\|_{C(0,1)}\end{aligned}$$

and with a similar argument

$$\begin{aligned}&\left|\big(\nabla E_{\mathcal{C}}^{\epsilon}(w,v)-\nabla E_\epsilon(w,v),\nabla E_\epsilon(w,v)\big)_{L^2(R_\epsilon)}\right|\\ &\quad\leqslant\left\|\frac{\partial E_{\mathcal{C}}^{\epsilon}(w,v)}{\partial s}-\frac{\partial E_\epsilon(w,v)}{\partial s}\right\|_{L^2(R_\epsilon)}\left\|\frac{\partial E_\epsilon(w,v)}{\partial s}\right\|_{L^2(\widetilde{R}_\epsilon)}\leqslant C\epsilon^N\|w\|_{C^1(\overline{\Omega})}\|v\|_{C^1(0,1)},\end{aligned}$$

which proves the lemma. □

A.4. Some auxiliary lemmas

Denote by $p_0=(0,\mathbf{0})$, $p_1=(1,\mathbf{0})$, $\mathbf{0}\in\mathbb{R}^{N-1}$ and p a generic point in $\mathbb{R}^N$. $B(p,\rho)$ is the ball of radius ρ around p. Consider the following:

$$\begin{gathered}B_\rho^l(p)=B(p,l)\setminus\overline{B(p,\rho)},\quad\text{for }\rho<l,\\ D_\rho^L=B(p_0,\rho)\cap\Omega,\quad\text{for }0<\rho\leqslant l,\qquad D_\rho^R=B(p_1,\rho)\cap\Omega,\quad\text{for }0<\rho\leqslant l,\\ S_\rho^L=\Omega\cap B_\rho^l(p_0),\qquad S_\rho^R=\Omega\cap B_\rho^l(p_1),\\ \widetilde{\Omega}^L=\Omega\setminus\overline{B(p_0,l)},\qquad \widetilde{\Omega}^R=\Omega\setminus\overline{B(p_1,l)},\\ \widehat{\Gamma}_\rho^0=\big\{(s,y)\colon\ |s|^2+|y|^2=\rho,\ s<0\big\},\\ \widehat{\Gamma}_\rho^1=\big\{(s,y)\colon\ |s-1|^2+|y|^2=\rho,\ s>1\big\},\\ (\phi,\psi)_{L^2(\Omega)}=\int\limits_\Omega\phi\cdot\psi,\qquad(\phi,\psi)_{L^2(R_\epsilon)}=\int\limits_{L^2(R_\epsilon)}\phi\cdot\psi.\end{gathered}\tag{A.19}$$

For a function ψ_ϵ defined in Ω_ϵ, we write

$$\widehat{T}_\rho^0\psi_\epsilon=\frac{1}{|\widehat{\Gamma}_\rho^0|}\int\limits_{\widehat{\Gamma}_\rho^0}\psi_\epsilon,\qquad\widehat{T}_\rho^1\psi_\epsilon=\frac{1}{|\widehat{\Gamma}_\rho^1|}\int\limits_{\widehat{\Gamma}_\rho^1}\psi_\epsilon.$$

With this we have the following result.

Lemma A.5. *Let $r \geqslant 1$, T_ϵ^0, T_ϵ^1 as in* (A.2), $\widehat{T}_{\epsilon r}^0$, $\widehat{T}_{\epsilon r}^1$ *as above and $\psi_\epsilon \in H^1(\Omega)$. Then, there is a constant $C = C(N)$ such that*

$$\begin{aligned} \left|T_\epsilon^0\psi_\epsilon - \widehat{T}_{\epsilon r}^0\psi_\epsilon\right| &\leqslant C(N)\epsilon^{(-N+2)/2}\|\nabla\psi_\epsilon\|_{D_{\epsilon r}^L}, \\ \left|T_\epsilon^1\psi_\epsilon - \widehat{T}_{\epsilon r}^1\psi_\epsilon\right| &\leqslant C(N)\epsilon^{(-N+2)/2}\|\nabla\psi_\epsilon\|_{D_{\epsilon r}^R}. \end{aligned} \tag{A.20}$$

Proof. We only prove the inequality for $|T_\epsilon^0\psi_\epsilon - \widehat{T}_{\epsilon r}^0\psi_\epsilon|$. The other is similar. Observe that $\epsilon D_r^L = D_{\epsilon r}^L$ and $\epsilon\Gamma_1^0 = \Gamma_\epsilon^0$. If $\psi_\epsilon \in H^1(\Omega)$, we define $\overline{\psi}_\epsilon(x,y) = \psi_\epsilon(\epsilon x, \epsilon y)$ for $(x,y) \in D_r^L$ and also $a_\epsilon = \frac{1}{|D_{\epsilon r}^L|}\int_{D_{\epsilon r}^L}\psi_\epsilon = \frac{1}{|D_r^L|}\int_{D_r^L}\overline{\psi}_\epsilon$. Thus,

$$\begin{aligned} \left|a_\epsilon - T_\epsilon^0\psi_\epsilon\right| &= \frac{1}{|\Gamma_\epsilon^0|}\left|\int_{\Gamma_\epsilon^0}\left(a_\epsilon - \psi_\epsilon(0,y)\right)dy\right| = \left|\frac{1}{|\Gamma_1^0|}\int_{\Gamma_1^0}\left(a_\epsilon - \overline{\psi}_\epsilon(0,y)\right)dy\right| \\ &\leqslant \frac{1}{|\Gamma_1^0|}\int_{\Gamma_1^0}|a_\epsilon - \overline{\psi}_\epsilon| \leqslant C\left\|a_\epsilon - \overline{\psi}_\epsilon(0,\cdot)\right\|_{L^2(\Gamma_1^0)} \\ &\leqslant C\|a_\epsilon - \overline{\psi}_\epsilon\|_{H^1(D_r^L)} = C\|a_\epsilon - \overline{\psi}_\epsilon\|_{L^2(D_r^L)} + C\|\nabla\overline{\psi}_\epsilon\|_{L^2(D_r^L)}. \end{aligned}$$

Now, the Poincaré inequality $\|a_\epsilon - \overline{\psi}_\epsilon\|_{L^2(D_r^L)} \leqslant C\|\nabla\overline{\psi}_\epsilon\|_{L^2(D_r^L)}$, implies that $|a_\epsilon - T_\epsilon^0\psi_\epsilon| \leqslant C\|\nabla\overline{\psi}_\epsilon\|_{L^2(D_r^L)}$. Since $\|\nabla\overline{\psi}_\epsilon\|_{L^2(D_r^L)}^2 = \epsilon^{-N+2}\|\nabla\psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}^2$ we conclude that $|a_\epsilon - T_\epsilon^0\psi_\epsilon| \leqslant \epsilon^{(-N+2)/2}\|\nabla\psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}$.

Following the same reasoning as above we obtain $|a_\epsilon - \widehat{T}_\epsilon^0\psi_\epsilon| \leqslant \epsilon^{(-N+2)/2}\|\nabla\psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}$ and therefore

$$\left|T_\epsilon^0\psi_\epsilon - \widehat{T}_\epsilon^0\psi_\epsilon\right| \leqslant \left|T_\epsilon^0\psi_\epsilon - a_\epsilon\right| + \left|a_\epsilon - \widehat{T}_\epsilon^0\psi_\epsilon\right|.$$

Hence, using $|T_\epsilon^0\psi_\epsilon - a_\epsilon|$ and $|a_\epsilon - \widehat{T}_\epsilon^0\psi_\epsilon|$ in the previous inequality, we conclude the proof of the lemma. □

Lemma A.6. *There is a constant $C = C(N)$ such that, if $\psi_\epsilon \in H^1(\Omega)$ then,*

$$\left(T_\epsilon^0\psi_\epsilon - \widehat{T}_r^0\psi_\epsilon\right)^2 + \left(T_\epsilon^1\psi_\epsilon - \widehat{T}_r^1\psi_\epsilon\right)^2 \leqslant \begin{cases} C(2)|\ln\epsilon|\|\psi_\epsilon\|_{H^1(\Omega)}^2, & \text{for } N = 2, \\ C(N)\epsilon^{2-N}\|\psi_\epsilon\|_{H^1(\Omega)}^2, & \text{for } N > 2. \end{cases} \tag{A.21}$$

Proof. We prove the lemma for $N > 2$. For the case $N = 2$, we refer to [4]. For $i = 0, 1$, consider the operators $\widehat{T}_\rho^i\psi_\epsilon$, as above, $0 < \rho \leqslant r$.

We have that

$$\|\nabla\psi_\epsilon\|_\Omega^2 \geqslant \|\nabla\psi_\epsilon\|_{D_r^L}^2 + \|\nabla\psi_\epsilon\|_{D_r^R}^2 = \|\nabla\psi_\epsilon\|_{S_{\epsilon r}^L}^2 + \|\nabla\psi_\epsilon\|_{D_{\epsilon r}^L}^2 + \|\nabla\psi_\epsilon\|_{S_{\epsilon r}^R}^2 + \|\nabla\psi_\epsilon\|_{D_{\epsilon r}^R}^2.$$

But

$$\begin{aligned}\|\nabla\psi_\epsilon\|^2_{S^L_{\epsilon r}} &\geqslant \min\bigl\{\|\nabla\chi_\epsilon\|^2_{S^L_{\epsilon r}}:\ \widehat{T}_l^{\,0}\chi_\epsilon=\widehat{T}_r^{\,0}\psi_\epsilon,\ \widehat{T}^{\,0}_{\epsilon r}\chi_\epsilon=\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon\bigr\}\\ &=\bigl(\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon-\widehat{T}_r^{\,0}\psi_\epsilon\bigr)^2\min\bigl\{\|\nabla\tilde\chi_\epsilon\|^2_{S^L_{\epsilon r}}:\ \widehat{T}_r^{\,0}\tilde\chi_\epsilon=0,\ \widehat{T}^{\,0}_{\epsilon r}\tilde\chi_\epsilon=1\bigr\}\\ &=\bigl(\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon-\widehat{T}_r l^0\psi_\epsilon\bigr)^2C(N)\epsilon^{N-2}\bigl(1+o(1)\bigr),\end{aligned}$$

where we have used that the minimum is attained when $\tilde\chi_\epsilon$ is the solution of

$$\begin{cases}-\Delta\tilde\chi_\epsilon=0, & S^L_{\epsilon r},\\ \widehat{T}_r^{\,0}\tilde\chi_\epsilon=0, & \widehat{T}^{\,0}_{\epsilon r}\tilde\chi=1,\end{cases}$$

and $\min\{\|\nabla\chi\|^2_{S^L_\eta}:\ \widehat{T}_r^{\,0}\chi=0,\ \widehat{T}_\eta^{\,0}\chi=1\}=C(N)\frac{\eta^{N-2}}{1-\eta^{N-2}}$. From Lemma A.5, we have that

$$\|\nabla\psi_\epsilon\|^2_{D^L_{\epsilon r}}\geqslant C(N)\epsilon^{N-2}\bigl(T^0_\epsilon\psi_\epsilon-\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon\bigr)^2.$$

Therefore

$$\begin{aligned}\|\nabla\psi_\epsilon\|^2_{S^L_{\epsilon r}}+\|\nabla\psi_\epsilon\|^2_{D^L_{\epsilon r}} &\geqslant C(N)\bigl[\bigl(\widehat{T}_r^{\,0}\psi_\epsilon-\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon\bigr)^2+\bigl(\widehat{T}^{\,0}_{\epsilon r}\psi_\epsilon-T^0_\epsilon\psi_\epsilon\bigr)^2\bigr]\epsilon^{N-2}\bigl(1+o(1)\bigr)\\ &\geqslant\frac{1}{2}C(N)\bigl(\widehat{T}_r^{\,0}\psi_\epsilon-T^0_\epsilon\psi_\epsilon\bigr)^2\epsilon^{N-2}\bigl(1+o(1)\bigr).\end{aligned}$$

With a similar reasoning we obtain an estimate for $\|\nabla\psi_\epsilon\|^2_{S^R_{\epsilon r}}+\|\nabla\psi_\epsilon\|^2_{D^R_{\epsilon r}}$. This concludes the proof. □

Lemma A.7. *There is a constant $C=C(N)$ such that, if $\psi_\epsilon\in H^1(\Omega)$ then,*

$$\bigl|T^0_\epsilon\psi_\epsilon\bigr|+\bigl|T^1_\epsilon\psi_\epsilon\bigr|\leqslant\begin{cases}C(2)|\ln\epsilon|^{1/2}\|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N=2,\\ C\epsilon^{(2-N)/2}\|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N>2.\end{cases}\tag{A.22}$$

Proof. The proof of this lemma follows from the previous and from the fact that $|\widehat{T}_r^{\,0}\psi_\epsilon|+|\widehat{T}_r^{1}\psi_\epsilon|\leqslant C\|\psi_\epsilon\|_{H^1(\Omega)}$. □

A.5. *Proof of Proposition 2.7*

In Appendix A.5 we will provide a proof of Proposition 2.7. We need to prove first some preliminary results.

Let $f_\epsilon\in U^p_\epsilon$ and define the functions $u_\epsilon\in H^1(\Omega_\epsilon)$, $w_\epsilon\in H^1(\Omega)$ and $v_\epsilon\in H^1(0,1)$ as the solutions of the following linear elliptic problems:

$$\begin{cases}-\Delta u_\epsilon+u_\epsilon=f_\epsilon, & \text{in } \Omega_\epsilon,\\ \frac{\partial u_\epsilon}{\partial n}=0, & \text{in } \partial\Omega_\epsilon,\end{cases}\tag{A.23}$$

$$\begin{cases}-\Delta w_\epsilon+w_\epsilon=f_\epsilon, & \text{in } \Omega,\\ \frac{\partial w_\epsilon}{\partial n}=0, & \text{in } \partial\Omega,\end{cases}\tag{A.24}$$

$$\begin{cases} -\frac{1}{g}(g(v_\epsilon)_s)_s + v_\epsilon = M_\epsilon f_\epsilon, & \text{in } (0,1), \\ v_\epsilon(0) = w_\epsilon(0), \qquad v_\epsilon(1) = w_\epsilon(1). \end{cases} \tag{A.25}$$

We have the following fundamental result:

Proposition A.8. *Let $p > N$. There exists a constant $C > 0$ such that for any $f_\epsilon \in U_\epsilon^p$ with $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$, we have*

$$\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)} + \|u_\epsilon - v_\epsilon\|^2_{H^1(R_\epsilon)} \leqslant \begin{cases} C\epsilon^2|\ln\epsilon|, & \text{for } N = 2, \\ C\epsilon^N, & \text{for } N > 2. \end{cases} \tag{A.26}$$

Proof. Notice first that since $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$, we have that $\|f_\epsilon\|_{L^p(\Omega)} \leqslant 1$, which implies that, since $p > N$, $w_\epsilon \in W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ and $\|w_\epsilon\|_{C^1(\overline{\Omega})} \leqslant C$ with some constant C independent of ϵ. With a similar regularity argument, we can easily show that $v_\epsilon \in C^1([0,1])$ and $\|v_\epsilon\|_{C^1([0,1])} \leqslant C$, with C independent of ϵ.

The solutions of the three problems (A.23)–(A.25) can be obtained by a minimization procedure. That is, if we define

$$\lambda_\epsilon = \min_{\phi\in H^1(\Omega_\epsilon)} \left\{ \frac{1}{2}\int_{\Omega_\epsilon} \left(|\nabla\phi|^2 + \phi^2\right) dx - \int_{\Omega_\epsilon} f_\epsilon \phi \, dx \right\},$$

$$\mu_\epsilon = \min_{\phi\in H^1(\Omega)} \left\{ \frac{1}{2}\int_{\Omega} \left(|\nabla\phi|^2 + \phi^2\right) dx - \int_{\Omega} f_\epsilon \phi \, dx \right\},$$

$$\tau_\epsilon = \min\left\{ \frac{1}{2}\int_0^1 g|\phi'|^2 + g\phi^2 - \int_0^1 g\phi\colon\ \phi \in H^1(0,1),\ \phi(0) = w_\epsilon(p_0),\ \phi(1) = w_\epsilon(p_1) \right\},$$

then λ_ϵ, μ_ϵ and τ_ϵ are attained in u_ϵ, w_ϵ, v_ϵ, respectively, and only there.

Let us first find a relationship among the three values λ_ϵ, μ_ϵ and τ_ϵ.

If we take the function $\varphi_\epsilon(x) = E_C^\epsilon(w_\epsilon, v_\epsilon)$, and denote by $\tilde{v}_\epsilon$ the component of φ_ϵ in R_ϵ, we obtain that

$$\begin{aligned} \lambda_\epsilon &\leqslant \frac{1}{2}\int_{\Omega_\epsilon} \left(|\nabla\varphi_\epsilon|^2 + \varphi_\epsilon^2\right) dx - \int_{\Omega_\epsilon} f_\epsilon \varphi_\epsilon \, dx \\ &= \frac{1}{2}\int_{\Omega} \left(|\nabla w_\epsilon|^2 + w_\epsilon^2\right) dx - \int_{\Omega} f_\epsilon w_\epsilon \, dx + \frac{1}{2}\int_{R_\epsilon} \left(|\nabla\tilde{v}_\epsilon|^2 + \tilde{v}_\epsilon^2\right) dx - \int_{R_\epsilon} f_\epsilon \tilde{v}_\epsilon \, dx \\ &= \mu_\epsilon + \frac{1}{2}\int_{R_\epsilon} \left(|\nabla\tilde{v}_\epsilon|^2 + \tilde{v}_\epsilon^2\right) dx - \int_{R_\epsilon} f_\epsilon \tilde{v}_\epsilon \, dx. \end{aligned}$$

It follows from Lemma A.4 (A.14) and (A.15)

$$\frac{1}{2}\int_{R_\epsilon}\big(|\nabla\tilde v_\epsilon|^2+\tilde v_\epsilon^2\big)\,dx\leqslant\frac{1}{2}\int_{R_\epsilon}\big(|\nabla v_\epsilon|^2+v_\epsilon^2\big)\,dx+C\epsilon^N\|w_\epsilon\|_{C^1(\Omega)}\|v_\epsilon\|_{C^1(\Omega)}$$
$$\leqslant\epsilon^{N-1}\frac{1}{2}\int_0^1\big(g(s)|v_\epsilon'|^2+g(s)v_\epsilon^2\big)\,ds+C\epsilon^N,$$

where we have used that $\|w_\epsilon\|_{C^1(\Omega)}$ and $\|v_\epsilon\|_{C^1(0,1)}$ are uniformly bounded. Moreover,

$$\int_{R_\epsilon} f_\epsilon\tilde v_\epsilon\,dx=\int_{R_\epsilon}M_\epsilon(f_\epsilon)v_\epsilon\,dx+\int_{R_\epsilon}M_\epsilon(f_\epsilon)(\tilde v_\epsilon-v_\epsilon)\,dx$$
$$=\epsilon^{N-1}\int_0^1 g(s)M_\epsilon(f_\epsilon)v_\epsilon\,ds+\int_{\widetilde R_\epsilon}M_\epsilon(f_\epsilon)(\tilde v_\epsilon-v_\epsilon)\,dx.$$

But

$$\left|\int_{\widetilde R_\epsilon}M_\epsilon(f_\epsilon)(\tilde v_\epsilon-v_\epsilon)\,dx\right|\leqslant\|M_\epsilon f_\epsilon\|_{L^2(\widetilde R_\epsilon)}\|\tilde v_\epsilon-v_\epsilon\|_{L^2(\widetilde R_\epsilon)}\leqslant C\|f_\epsilon\|_{L^2(\widetilde R_\epsilon)}\epsilon^{\frac{N+2}{2}}\|w_\epsilon\|_{C^1(\overline{\Omega})}$$

and by Hölder,

$$\|f_\epsilon\|_{L^2(\widetilde R_\epsilon)}\leqslant\|f_\epsilon\|_{L^p(\widetilde R_\epsilon)}|\widetilde R_\epsilon|^{\frac12-\frac1p}\leqslant\epsilon^{-\frac{N-1}{p}}\|f_\epsilon\|_{L^p(\widetilde R_\epsilon)}\epsilon^{\frac{N-1}{p}}\big(\epsilon^N\big)^{\frac12-\frac1p}\leqslant C\epsilon^{\frac N2-\frac1p}.$$

This implies

$$\left|\int_{\widetilde R_\epsilon}M_\epsilon(f_\epsilon)(\tilde v_\epsilon-v_\epsilon)\,dx\right|\leqslant C\epsilon^{N+1-\frac1p}.$$

In particular, we obtain the following upper bounds for λ_ϵ

$$\lambda_\epsilon\leqslant\mu_\epsilon+\epsilon^{N-1}\tau_\epsilon+C\epsilon^N. \tag{A.27}$$

To obtain the lower bounds, we proceed as follows. From the definition of λ_ϵ we have

$$\lambda_\epsilon=\frac12\int_{\Omega_\epsilon}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int_{\Omega_\epsilon}f_\epsilon u_\epsilon\,dx$$
$$=\frac12\int_{\Omega}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int_{\Omega}f_\epsilon u_\epsilon\,dx+\frac12\int_{R_\epsilon}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int_{R_\epsilon}f_\epsilon u_\epsilon\,dx. \tag{A.28}$$

But

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int\limits_{\Omega} f_\epsilon u_\epsilon\,dx\\
&\quad=\frac{1}{2}\int\limits_{\Omega}\big(|\nabla u_\epsilon-\nabla w_\epsilon+\nabla w_\epsilon|^2+(u_\epsilon-w_\epsilon+w_\epsilon)^2\big)\,dx-\int\limits_{\Omega} f_\epsilon(u_\epsilon-w_\epsilon+w_\epsilon)\,dx\\
&\quad=\frac{1}{2}\int\limits_{\Omega}\big(|\nabla w_\epsilon|^2+w_\epsilon^2\big)\,dx-\int\limits_{\Omega} f_\epsilon w_\epsilon\,dx+\frac{1}{2}\int\limits_{\Omega}\big(|\nabla u_\epsilon-\nabla w_\epsilon|^2+(u_\epsilon-w_\epsilon)^2\big)\,dx\\
&\qquad+\int\limits_{\Omega}(u_\epsilon-w_\epsilon)w_\epsilon+\int\limits_{\Omega}(\nabla u_\epsilon-\nabla w_\epsilon)\nabla w_\epsilon-\int\limits_{\Omega} f_\epsilon(u_\epsilon-w_\epsilon)\\
&\quad=\frac{1}{2}\int\limits_{\Omega}\big(|\nabla w_\epsilon|^2+w_\epsilon^2\big)\,dx-\int\limits_{\Omega} f_\epsilon w_\epsilon\,dx+\frac{1}{2}\int\limits_{\Omega}\big(|\nabla u_\epsilon-\nabla w_\epsilon|^2+(u_\epsilon-w_\epsilon)^2\big)\,dx,
\end{aligned}
\tag{A.29}
$$

where in the last equality we have used the integration by parts of $\int_\Omega(\nabla u_\epsilon-\nabla w_\epsilon)\nabla w_\epsilon$ and the fact that w_ϵ is the solution of the elliptic problem (A.24) in Ω, that is,

$$
\int\limits_{\Omega}(\nabla u_\epsilon-\nabla w_\epsilon)\nabla w_\epsilon=\int\limits_{\partial\Omega}(u_\epsilon-w_\epsilon)\frac{\partial w_\epsilon}{\partial n}-\int\limits_{\Omega}(u_\epsilon-w_\epsilon)\Delta w_\epsilon=\int\limits_{\Omega}(u_\epsilon-w_\epsilon)(f_\epsilon-w_\epsilon).
$$

Also we have

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{R_\epsilon}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int\limits_{R_\epsilon} f_\epsilon u_\epsilon\,dx\\
&\quad=\frac{1}{2}\int\limits_{R_\epsilon}\big(|\nabla u_\epsilon-\nabla v_\epsilon+\nabla v_\epsilon|^2+(u_\epsilon-v_\epsilon+v_\epsilon)^2\big)\,dx-\int\limits_{R_\epsilon} f_\epsilon v_\epsilon\,dx-\int\limits_{R_\epsilon} f_\epsilon(u_\epsilon-v_\epsilon)\\
&\quad=\frac{1}{2}\int\limits_{R_\epsilon}\big(|\nabla u_\epsilon-\nabla v_\epsilon|^2+(u_\epsilon-v_\epsilon)^2\big)+\frac{1}{2}\int\limits_{R_\epsilon}\big(|\nabla v_\epsilon|^2+v_\epsilon^2\big)\\
&\qquad+\int\limits_{R_\epsilon}(\nabla u_\epsilon-\nabla v_\epsilon)\nabla v_\epsilon+\int\limits_{R_\epsilon}(u_\epsilon-v_\epsilon)v_\epsilon-\int\limits_{R_\epsilon} f_\epsilon v_\epsilon\,dx-\int\limits_{R_\epsilon} f_\epsilon(u_\epsilon-v_\epsilon).
\end{aligned}
\tag{A.30}
$$

But

$$
\begin{aligned}
\int\limits_{R_\epsilon}(\nabla u_\epsilon-\nabla v_\epsilon)\nabla v_\epsilon&=\int\limits_{R_\epsilon}\left(\frac{\partial u_\epsilon}{\partial s}-\frac{\partial v_\epsilon}{\partial s}\right)\frac{\partial v_\epsilon}{\partial s}\\
&=\int\limits_{R_\epsilon}\left(\frac{\partial u_\epsilon}{\partial s}-\frac{\partial M_\epsilon u_\epsilon}{\partial s}\right)\frac{\partial v_\epsilon}{\partial s}+\int\limits_{R_\epsilon}\left(\frac{\partial M_\epsilon u_\epsilon}{\partial s}-\frac{\partial v_\epsilon}{\partial s}\right)\frac{\partial v_\epsilon}{\partial s}=I_1+I_2.
\end{aligned}
\tag{A.31}
$$

First we analyze I_1. Observe first that with the change of variables $y = \epsilon L_s(z)$ (see (2.1) and (2.2) for the definition of L_s and L), we get

$$\frac{1}{|\Gamma_\epsilon^s|}\int\limits_{\Gamma_\epsilon^s} u_\epsilon(s,y)\,dy = \int\limits_{B(0,1)} u_\epsilon\big(s,\epsilon L_s(z)\big)\frac{1}{|\Gamma_1^s|}J_{L_s}(z)\,dz.$$

This implies that

$$\begin{aligned}
\frac{dM_\epsilon u_\epsilon}{ds}(s) &= \frac{d}{ds}\int\limits_{B(0,1)} u_\epsilon\big(s,\epsilon L_s(z)\big)\frac{J_{L_s}(z)}{|\Gamma_1^s|}\,dz \\
&= \int\limits_{B(0,1)} \frac{\partial u_\epsilon}{\partial s}\big(s,\epsilon L_s(z)\big)\frac{J_{L_s}(z)}{|\Gamma_1^s|}\,dz + \int\limits_{B(0,1)} \nabla_y u_\epsilon\big(s,\epsilon L_s(z)\big)\epsilon\frac{\partial}{\partial s}\big(L_s(z)\big)\frac{J_{L_s}(z)}{|\Gamma_1^s|}\,dz \\
&\quad + \int\limits_{B(0,1)} u_\epsilon\big(s,\epsilon L_s(z)\big)\frac{\partial}{\partial s}\big(J_{L_s}(z)/\big|\Gamma_1^s\big|\big)\,dz \\
&= K_1 + K_2 + K_3.
\end{aligned}$$

Undoing the change of variables in K_1, we get

$$K_1 = \frac{1}{\epsilon^{N-1}}\int\limits_{\Gamma_\epsilon^s}\frac{\partial u_\epsilon}{\partial s}(s,y)\,dy.$$

Moreover, using that $|\frac{\partial L_s(z)}{\partial s}| \leqslant C$ and undoing the change of variables, we have

$$|K_2| \leqslant C\epsilon\int\limits_{B(0,1)}\big|\nabla_y u_\epsilon\big(s,\epsilon L_s(z)\big)\big|J_{L_s}(z)\,dz \leqslant C\frac{\epsilon}{|\Gamma_\epsilon^s|}\int\limits_{\Gamma_\epsilon^s}\big|\nabla_y u_\epsilon(s,y)\big|\,dy.$$

Now,

$$\begin{aligned}
K_3 &= \int\limits_{B(0,1)} u_\epsilon\big(s,\epsilon L_s(z)\big)\frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z)\,dz \\
&= \int\limits_{B(0,1)} \big(u_\epsilon\big(s,\epsilon L_s(z)\big) - (M_\epsilon u_\epsilon)(s)\big)\frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z)\,dz \\
&\quad + (M_\epsilon u_\epsilon)(s)\int\limits_{B(0,1)}\frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z)\,dz.
\end{aligned}$$

But

$$\int\limits_{B(0,1)}\frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z)\,dz = \frac{d}{ds}\left(\frac{1}{|\Gamma_1^s|}\int\limits_{B(0,1)} J_{L_s}(z)\,dz\right) = 0$$

because $\int_{B(0,1)} J_{L_s}(z)\,dz = |\Gamma_1^s|$.

Similarly undoing the change of variables and using that $|\frac{\partial(J_{L_s}(z)/|\Gamma_1^s|)}{\partial s}| \leqslant C$ we have

$$|K_3| \leqslant C \frac{1}{|\Gamma_\epsilon^s|} \int_{\Gamma_\epsilon^s} \left|u_\epsilon(s,y) - M_\epsilon u_\epsilon(s)\right| dy.$$

Putting all the estimates together, we get

$$\left|\frac{dM_\epsilon u_\epsilon}{ds}(s) - M_\epsilon\left(\frac{\partial u_\epsilon}{\partial s}\right)\right| \leqslant C \frac{1}{|\Gamma_\epsilon^s|}\left(\int_{\Gamma_\epsilon^s} \left|u_\epsilon(s,y) - M_\epsilon u_\epsilon(s)\right| dy + \epsilon \int_{\Gamma_\epsilon^s} \left|\nabla_y u_\epsilon(s,y)\right| dy\right).$$

Now,

$$|I_1| \leqslant C\epsilon \|v_\epsilon'\|_{L^2(R_\epsilon)} \|\nabla_y u_\epsilon\|_{L^2(R_\epsilon)} + C\|v_\epsilon'\|_{L^2(R_\epsilon)} \|u_\epsilon - M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)}.$$

But by Poincaré inequality,

$$\|u_\epsilon - M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)} \leqslant C\epsilon \|\nabla_y u_\epsilon\|_{L^2(R_\epsilon)}$$

which implies that

$$|I_1| \leqslant C\epsilon \|v_\epsilon'\|_{L^2(R_\epsilon)} \|\nabla_y u_\epsilon\|_{L^2(R_\epsilon)}.$$

But we obviously have uniform estimates of $\|v_\epsilon\|_{H^1(0,1)}$. Hence, we have that

$$|I_1| \leqslant C\epsilon^{\frac{N+1}{2}} \|\nabla_y u_\epsilon\| \leqslant C\epsilon^{N+1} + \frac{1}{4}\|\nabla_y u_\epsilon\|^2_{L^2(R_\epsilon)}$$

and observe that $\|\nabla_y u_\epsilon\|^2_{L^2(R_\epsilon)} \leqslant \|\nabla u_\epsilon - \nabla v_\epsilon\|^2_{L^2(R_\epsilon)}$, which implies

$$|I_1| \leqslant C\epsilon^{N+1} + \frac{1}{4}\|\nabla u_\epsilon - \nabla v_\epsilon\|^2_{L^2(R_\epsilon)}.$$

On the other hand, observe that

$$\begin{aligned}
I_2 &= \epsilon^{N-1} \int_0^1 \left((M_\epsilon u_\epsilon)' - v_\epsilon'\right) g(s) v_\epsilon' \\
&= -\epsilon^{N-1} \int_0^1 (M_\epsilon u_\epsilon - v_\epsilon)\left(g(s)v_\epsilon'\right)' + \left[(M_\epsilon u_\epsilon - v_\epsilon)\left(g(s)v_\epsilon'\right)\right]_0^1 \\
&= -\epsilon^{N-1} \int_0^1 (M_\epsilon u_\epsilon - v_\epsilon)\left[g(s)v_\epsilon - g(s)M_\epsilon f_\epsilon\right] + \epsilon^{N-1}\left[(M_\epsilon u_\epsilon - v_\epsilon)\left(g(s)v_\epsilon'\right)\right]_0^1 \\
&= -\int_{R_\epsilon} (u_\epsilon - v_\epsilon)v_\epsilon + \int_{R_\epsilon} (u_\epsilon - v_\epsilon) f_\epsilon + \int_{R_\epsilon} (M_\epsilon u_\epsilon - u_\epsilon) f_\epsilon + \epsilon^{N-1}\left[(M_\epsilon u_\epsilon - v_\epsilon)\left(g(s)v_\epsilon'\right)\right]_0^1.
\end{aligned} \tag{A.32}$$

From which we obtain

$$\int_{R_\epsilon}\big(|\nabla u_\epsilon|^2+u_\epsilon^2\big)\,dx-\int_{R_\epsilon} f_\epsilon u_\epsilon\,dx=\int_{R_\epsilon}\big(|\nabla v_\epsilon|^2+v_\epsilon^2\big)\,dx-\int_{\Omega} f_\epsilon v_\epsilon\,dx$$
$$+\int_{R_\epsilon}\big(|\nabla u_\epsilon-\nabla v_\epsilon|^2+(u_\epsilon-v_\epsilon)^2\big)\,dx+\kappa(\epsilon)+\eta(\epsilon)+I_1, \tag{A.33}$$

where $\kappa(\epsilon)=\int_{R_\epsilon}(M_\epsilon u_\epsilon-u_\epsilon)f_\epsilon$ and $\eta(\epsilon)=\epsilon^{N-1}[(M_\epsilon u_\epsilon-v_\epsilon)(g(s)v_\epsilon')]_0^1$.

From (A.5) (Lemma A.3), we obtain that

$$\begin{aligned}\big|\kappa(\epsilon)\big| &\leqslant \bigg|\int_{R_\epsilon}(u_\epsilon-M_\epsilon u_\epsilon)f_\epsilon\bigg|\leqslant \|u_\epsilon-M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)}\|f_\epsilon\|_{L^2(R_\epsilon)}\\
&\leqslant C\epsilon^{\frac{N-1}{2}}\|u_\epsilon-M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)}\big(\epsilon^{\frac{1-N}{2}}\|f_\epsilon\|_{L^2(R_\epsilon)}\big)\\
&\leqslant C\epsilon^{\frac{N+1}{2}}\|\nabla_y u\|_{L^2(R_\epsilon)}\leqslant C\epsilon^{N+1}+\frac14\|\nabla_y u\|^2_{L^2(R_\epsilon)}\\
&\leqslant C\epsilon^{N+1}+\frac14\|\nabla u-\nabla v_\epsilon\|^2_{L^2(R_\epsilon)},\end{aligned}\tag{A.34}$$

where we have used that $\epsilon^{\frac{1-N}{2}}\|f_\epsilon\|_{L^2(R_\epsilon)}\leqslant C\epsilon^{\frac{1-N}{p}}\|f_\epsilon\|_{L^p(R_\epsilon)}\leqslant C$.

We estimate now $\eta(\epsilon)$. For this, note first that $v_\epsilon(0)=w_\epsilon(0,0)$ and $v_\epsilon(1)=w_\epsilon(1,0)$. In particular,

$$\begin{aligned}\big|T_\epsilon^0(w_\epsilon-v_\epsilon)\big|&=\frac{1}{|\Gamma_\epsilon^0|}\bigg|\int_{\Gamma_\epsilon^0}\big(w_\epsilon(0,y)-w_\epsilon(0,0)\big)\,dy\bigg|\leqslant\frac{C}{|\Gamma_\epsilon^0|}\int_{\Gamma_\epsilon^0}|y|\,dy\,\|w_\epsilon\|_{C^1(\overline{\Omega}_0)}\\
&\leqslant C\epsilon\|w_\epsilon\|_{C^1(\overline{\Omega}_0)}\end{aligned}$$

and, similarly, we obtain

$$\big|T_\epsilon^1(w_\epsilon-v_\epsilon)\big|\leqslant C\epsilon\|w_\epsilon\|_{C^1(\overline{\Omega}_0)}.$$

Hence,

$$\begin{aligned}\big|\eta(\epsilon)\big|&\leqslant C\epsilon^{N-1}\big(\big|T_\epsilon^1(u_\epsilon)-v_\epsilon(1)\big|+\big|T_\epsilon^0(u_\epsilon)-v_\epsilon(0)\big|\big)\|v_\epsilon\|_{C^1([0,1])}\\
&\leqslant C\epsilon^{N-1}\big(\big|T_\epsilon^1(u_\epsilon-w_\epsilon)\big|+\big|T_\epsilon^1(w_\epsilon)-v_\epsilon(1)\big|+\big|T_\epsilon^0(u_\epsilon-w_\epsilon)\big|\\
&\quad+\big|T_\epsilon^0(w_\epsilon)-v_\epsilon(0)\big|\big)\|v_\epsilon\|_{C^1([0,1])}\\
&\leqslant C\epsilon^{N-1}\big(\big|T_\epsilon^1(u_\epsilon-w_\epsilon)\big|+\big|T_\epsilon^0(u_\epsilon-w_\epsilon)\big|\big)\|v_\epsilon\|_{C^1([0,1])}+C\epsilon^N\|v_\epsilon\|_{C^1([0,1])}\|w_\epsilon\|_{C^1(\overline{\Omega}_0)}.\end{aligned}$$

But using the fact that $\|v_\epsilon\|_{C^1([0,1])}$, $\|w_\epsilon\|_{C^1(\overline{\Omega}_0)} \leqslant C$ and applying Lemma A.7 with $\psi_\epsilon = u_\epsilon - w_\epsilon$, we get

$$\begin{aligned}\left|\eta(\epsilon)\right| &\leqslant C(N)\epsilon^{N-1}\left[\left|T_\epsilon^0(u_\epsilon - v_\epsilon)\right| + \left|T_\epsilon^1(u_\epsilon - v_\epsilon)\right|\right] + C\epsilon^N \\ &\leqslant C\big(\Theta_N(\epsilon)\big)^{1/2}\|u_\epsilon - w_\epsilon\|_{H^1(\Omega)} + C\epsilon^N,\end{aligned}$$

where

$$\Theta_N(\epsilon) = \begin{cases} \epsilon^2|\ln\epsilon|, & \text{for } N = 2,\\ \epsilon^N, & \text{for } N > 2,\end{cases}$$

and therefore

$$\left|\eta(\epsilon)\right| \leqslant C\Theta_N(\epsilon) + \frac{1}{4}\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)},$$

where we have used that $\Theta_N(\epsilon) \geqslant \epsilon^N$. Putting together all the estimates, we obtain

$$\begin{aligned}&\int\limits_{R_\epsilon}\big(|\nabla u_\epsilon|^2 + u_\epsilon^2\big)\,dx - \int\limits_{R_\epsilon} f_\epsilon u_\epsilon\,dx \\ &\quad \geqslant \int\limits_{R_\epsilon}\big(|\nabla v_\epsilon|^2 + v_\epsilon^2\big)\,dx - \int\limits_{\Omega} f_\epsilon v_\epsilon\,dx - \frac{1}{2}\|u_\epsilon - v_\epsilon\|^2_{H^1(R_\epsilon)} - \frac{1}{2}\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)} \\ &\quad\quad - C(N)\Theta_N(\epsilon).\end{aligned} \tag{A.35}$$

Thus,

$$\lambda_\epsilon \geqslant \mu_\epsilon + \epsilon^{N-1}\tau_\epsilon + \frac{1}{2}\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)} + \frac{1}{2}\|u_\epsilon - v_\epsilon\|^2_{H^1(R_\epsilon)} - C\Theta_N(\epsilon).$$

Since we have obtained that $\lambda_\epsilon \leqslant \mu_\epsilon + \epsilon^{N-1}\tau_\epsilon + C\epsilon^N$, and $\epsilon^N \leqslant \Theta_N(\epsilon)$, then

$$\|u_\epsilon - w_\epsilon\|^2_{H^1(\Omega)} + \|u_\epsilon - v_\epsilon\|^2_{H^1(R_\epsilon)} \leqslant C\Theta_N(\epsilon).$$

This completes the proof of the proposition. □

Lemma A.9. *Let $\{f_\epsilon\}$ be a sequence such that $f_\epsilon \in U_\epsilon^p$ and $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$. Then, there are functions $f \in L^p(\Omega)$ and $h \in L_g^p(0,1)$ such that*

$$\int\limits_{\Omega} f_\epsilon w_\epsilon\,dx \longrightarrow \int\limits_{\Omega} f w\,dx \quad \text{and} \quad \frac{1}{\epsilon^{N-1}}\int\limits_{R_\epsilon} f_\epsilon v_\epsilon\,dx \longrightarrow \int\limits_0^1 g(s)h(s)v(s)\,ds$$

whenever $\|w_\epsilon - w\|_{L^{p'}(\Omega)} + \epsilon^{\frac{1-N}{p'}}\|v_\epsilon - v\|_{L^{p'}(R_\epsilon)} \xrightarrow{\epsilon\to 0} 0.$

Proof. Note that $\|f_\epsilon\|_{U_\epsilon} \leqslant 1$ means

$$\int_\Omega \big(f_\epsilon(x)\big)^p\,dx + \frac{1}{\epsilon^{N-1}}\int_{R_\epsilon}\big(f_\epsilon(x)\big)^p\,dx \leqslant 1.$$

It follows that, there exists $f \in L^p(\Omega)$ such that $f_\epsilon \to f$ weakly in $L^p(\Omega)$. Hence

$$\int_\Omega f_\epsilon w_\epsilon\,dx \longrightarrow \int_\Omega f w\,dx$$

whenever $\|w_\epsilon - w\|_{L^p(\Omega)} \to 0$.

Also note that, writing $x = (s, y)$ with $s \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$ and $\epsilon\tilde{y} = y$, we have that

$$\frac{1}{\epsilon^{N-1}}\int_{R_\epsilon}\big(f_\epsilon(x)\big)^p\,dx \leqslant \int_{R_1}\big(\tilde{f}_\epsilon(s,\tilde{y})\big)^p\,ds\,d\tilde{y} \leqslant 1,$$

where $\tilde{f}_\epsilon(s,\tilde{y}) = f_\epsilon(s,\epsilon\tilde{y})$. So, there exists $\tilde{h} \in L^p(R_1)$ such that $\tilde{f}_\epsilon \to \tilde{h}$ weakly in $L^p(R_1)$. Let us show that $\tilde{h}$ is independent of y. Note that, if $\tilde{\phi} \in C_0^\infty(R_1)$ then,

$$\int_{R_1}\tilde{f}_\epsilon(s,\tilde{y})\frac{\partial\tilde{\phi}}{\partial\tilde{y}_i}(s,\tilde{y})\,ds\,d\tilde{y} = \frac{\epsilon}{\epsilon^{N-1}}\int_{R_\epsilon} f_\epsilon(s,y)\frac{\partial\phi}{\partial y_i}(s,y)\,ds\,dy \xrightarrow{\epsilon\to 0} 0.$$

If follows that, for all $\tilde{\phi} \in C_0^\infty(R_1)$,

$$\int_{R_1}\tilde{h}\frac{\partial\tilde{\phi}}{\partial\tilde{y}_i}\,dx = 0.$$

Hence, $\tilde{h}(s,y) = g(s)h(s)$ for some $h \in L^2(0,1)$. Furthermore, if $\epsilon^{\frac{1-N}{p}}\|v_\epsilon - v\|_{L^p(R_\epsilon)} \to 0$ then,

$$\frac{1}{\epsilon^{N-1}}\int_{R_\epsilon} f_\epsilon(x)v_\epsilon(x)\,dx \longrightarrow \int_0^1 g(s)h(s)v(s)\,ds. \qquad \square$$

Now we show the following result:

Proposition A.10. *Let $p > N$ and consider a sequence $f_\epsilon \in U_\epsilon^p$ with $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$. Let $(f,h) \in L^p(\Omega) \times L_g^p(0,1)$ such that $f_\epsilon \to (f,h)$ weakly in the sense of Lemma* A.9. *Then*

$$\|u_\epsilon - w\|_{H^1(\Omega)} + \frac{1}{\epsilon^{(N-1)/2}}\|u_\epsilon - v\|_{H^1(R_\epsilon)} \to 0, \tag{A.36}$$

where u_ϵ, w and v are the solutions of the following problems,

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon, & \textit{in } \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = 0, & \textit{in } \partial\Omega_\epsilon, \end{cases} \qquad \begin{cases} -\Delta w + w = f, & \textit{in } \Omega, \\ \frac{\partial w}{\partial n} = 0, & \textit{in } \partial\Omega, \end{cases} \tag{A.37}$$

$$\begin{cases} -\frac{1}{g}(g v_s)_s + v = h, & \textit{in } (0,1), \\ v(0) = w(0), & v(1) = w(1). \end{cases} \tag{A.38}$$

Proof. If w_ϵ and v_ϵ are given by (A.24) and (A.25), respectively, and taking into account that $p > N \geqslant 2$ and $f_\epsilon \to f$ weakly in $L^p(\Omega)$ we easily obtain that $\|w_\epsilon - w\|_{C^0(\overline{\Omega})} \to 0$ and $\|w_\epsilon - w\|_{H^1(\Omega)} \to 0$. From Lemma A.8 we get that

$$\|u_\epsilon - w\|_{H^1(\Omega)} \to 0.$$

Moreover, since $w_\epsilon \to w$ in $C^0(\overline{\Omega})$ and $M_\epsilon f_\epsilon \to h$ weakly in $L^p_g(0,1)$ it is very simple to see that we have

$$\|v_\epsilon - v\|_{H^1(0,1)} \to 0$$

which implies that

$$\frac{1}{\epsilon^{N-1}} \|v_\epsilon - v\|^2_{H^1(R_\epsilon)} \to 0.$$

Hence, with this last statement and using Lemma A.8 we get

$$\frac{1}{\epsilon^{(N-1)/2}} \|u_\epsilon - v\|_{H^1(R_\epsilon)} \leqslant \frac{1}{\epsilon^{(N-1)/2}} \big(\|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)} + \|v_\epsilon - v\|_{H^1(R_\epsilon)} \big) \to 0$$

which proves the result. □

We obtain now a result on uniform $L^\infty(\Omega_\epsilon)$ bounds for the family of solutions $\{u_\epsilon\}_{\epsilon\in[0,1]}$ of problem (A.23). This result will show part (i) of Proposition 2.7.

Lemma A.11. *There exists a constant C independent of ϵ such that for all $f_\epsilon \in U^p_\epsilon$ with $p > N/2$ and $\|f_\epsilon\|_{U^p_\epsilon} \leqslant 1$ if u_ϵ is the solution of* (A.23) *then*

$$\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C. \tag{A.39}$$

Proof. Let us define the functions u^1_ϵ and u^2_ϵ as the solutions of the following problems

$$\begin{cases} -\Delta u^1_\epsilon + u^1_\epsilon = f_\epsilon|_{R_\epsilon}, & \text{in } \Omega_\epsilon, \\ \frac{\partial u^1_\epsilon}{\partial n} = 0, & \text{in } \partial\Omega_\epsilon, \end{cases} \tag{A.40}$$

and

$$\begin{cases} -\Delta u^2_\epsilon + u^2_\epsilon = f_\epsilon|_{\Omega}, & \text{in } \Omega_\epsilon, \\ \frac{\partial u^2_\epsilon}{\partial n} = 0, & \text{in } \partial\Omega_\epsilon. \end{cases} \tag{A.41}$$

We proceed in several steps.

Step 1. Let us show that for each compact set $K \subset \overline{\Omega} \setminus \{0, 1\}$ there is a constant C independent of ϵ such that

$$\|u_\epsilon\|_{L^\infty(K)} \leqslant \|u_\epsilon^1\|_{L^\infty(K)} + \|u_\epsilon^2\|_{L^\infty(K)} \leqslant C.$$

This follows easily with a cutoff function and an elementary bootstrap argument.

Step 2. Let us see now that there is a constant C independent of ϵ such that

$$\|u_\epsilon^1\|_{L^\infty(R_\epsilon)} \leqslant C\epsilon^{\frac{1-N}{p}}\|f\|_{L^p(R_\epsilon)} \leqslant \|f\|_{U_\epsilon^p}.$$

To prove this result first note that, from step 1, for any compact subset of $\overline{\Omega} \setminus \{0, 1\}$ we have that u_ϵ^1 is uniform bounded in $L^\infty(K)$. Hence, if for a small fixed $\delta > 0$, we define $\widetilde{R}_\epsilon = ([-\delta, 0] \times \Gamma_\epsilon^0) \cup R_\epsilon \cup ([1, 1+\delta] \times \Gamma_\epsilon^1)$, then, there exists a $k > 0$ such that $|u_\epsilon^1| \leqslant k$ in $\Gamma_{(-\delta,1+\delta)} = (\{-\delta\} \times \Gamma_\epsilon^0) \cup (\{1+\delta\} \times \Gamma_\epsilon^1)$.

Hence, if we define $\phi_\epsilon = (u_\epsilon^1 - k)^+$ in Ω_ϵ, after multiplying the equation by ϕ_ϵ and integrating by parts we have that

$$\int\limits_{\widetilde{R}_\epsilon} |\nabla\phi_\epsilon|^2 + \phi_\epsilon^2 \leqslant \int\limits_{\Omega_\epsilon} |\nabla\phi_\epsilon|^2 + \phi_\epsilon^2 = \int\limits_{R_\epsilon} (f_\epsilon - k)\phi_\epsilon \leqslant \int\limits_{R_\epsilon} |f_\epsilon|\phi_\epsilon. \tag{A.42}$$

Writing $x = (s, y)$ with $s \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$ and changing the variables (s, y) to $(s, \tilde{y})$ where to $\epsilon\tilde{y} = y$ we obtain from (A.42) that

$$\int\limits_{\widetilde{R}_1} |\nabla\tilde{\phi}_\epsilon|^2 + |\tilde{\phi}_\epsilon|^2 \leqslant \int\limits_{R_1} \tilde{f}\tilde{\phi}_\epsilon, \tag{A.43}$$

where $\tilde{\phi}_\epsilon(s, \tilde{y}) = \phi_\epsilon(s, \epsilon\bar{y})$ and $\tilde{f}_\epsilon(s, \tilde{y}) = f_\epsilon(s, \epsilon\bar{y})$. Proceeding exactly as in [8, Lemma B.1(iii)] we obtain that

$$\|\tilde{\phi}_\epsilon\|_{H^1(\widetilde{R}^1)}^2 \leqslant \|\tilde{f}_\epsilon\|_{L^p(R_1)}\|\tilde{\phi}_\epsilon\|_{L^{p'}(R_1)} \leqslant \|\tilde{f}_\epsilon\|_{L^p(R_1)}\|\tilde{\phi}_\epsilon\|_{L^{\frac{2N}{N-2}}(R_1)}|A_k|^{\frac{1}{p'}+\frac{1}{N}-\frac{1}{2}}, \tag{A.44}$$

where $A_k = \{(x, \tilde{y}) \in \widetilde{R}_1\colon u_\epsilon^1 > k\}$. From this we have

$$\|\tilde{\phi}_\epsilon\|_{H^1(\widetilde{R}^1)} \leqslant \|\tilde{f}_\epsilon\|_{L^p(R_1)}|A_k|^{\frac{1}{p'}+\frac{1}{N}-\frac{1}{2}}. \tag{A.45}$$

From (A.44) and (A.45) we have that

$$\|\tilde{\phi}_\epsilon\|_{L^1(\widetilde{R}_1)} \leqslant \|\tilde{\phi}_\epsilon\|_{L^{\frac{2N}{N-2}}(\widetilde{R}_1)}|A_k|^{\frac{N+2}{2N}} \leqslant C\|\tilde{\phi}_\epsilon\|_{H^1(\widetilde{R}_1)}|A_k|^{\frac{N+2}{2N}} \leqslant C\|\tilde{f}_\epsilon\|_{L^p(R_1)}|A_k|^{\frac{1}{p'}+\frac{2}{N}}.$$

Since, for $p > \frac{N}{2}$ we have that $\frac{1}{p'} + \frac{2}{N} > 1$, it follows from [34, Lemma 5.1] that

$$\|\phi_\epsilon\|_{L^\infty(\widetilde{R}_\epsilon)} = \|\tilde{\phi}_\epsilon\|_{L^\infty(\widetilde{R}_1)} \leqslant C\|\tilde{f}\|_{L^p(R_1)} = C\epsilon^{\frac{1-N}{p}}\|f_\epsilon\|_{L^p(R_\epsilon)}$$

with $C = C(R_1, N, p)$.

Step 3. We show that $\|u_\epsilon^2\|_{L^\infty(\Omega)} \leqslant C$.

After multiplying Eq. (A.41) by $\psi_\epsilon = (u_\epsilon^2 - k)^+$, $k > 0$, and integrating by parts we have that

$$\int_\Omega |\nabla\psi_\epsilon|^2 + |\psi_\epsilon|^2 \leqslant \int_{\Omega_\epsilon} |\nabla\psi_\epsilon|^2 + |\psi_\epsilon|^2 = \int_{\Omega_\epsilon} (f_\epsilon - k)\psi_\epsilon \leqslant \int_\Omega f_\epsilon\psi_\epsilon. \tag{A.46}$$

Proceeding exactly as in [8, Lemma B.1(iii)] we obtain that

$$\|\psi_\epsilon\|^2_{H^1(\Omega)} \leqslant \|f_\epsilon\|_{L^p(\Omega)}\|\psi_\epsilon\|_{L^{p'}(\Omega)} \leqslant \|f_\epsilon\|_{L^p(\Omega)}\|\psi_\epsilon\|_{L^{\frac{2N}{N-2}}(\Omega)}|A_k|^{\frac{1}{p'}+\frac{1}{N}-\frac{1}{2}}, \tag{A.47}$$

where $A_k = \{(x,\tilde{y}) \in \Omega\colon u_\epsilon^2 > k\}$. From this we have

$$\|\psi_\epsilon\|_{H^1(\Omega)} \leqslant \|f_\epsilon\|_{L^p(\Omega)}|A_k|^{\frac{1}{p'}+\frac{1}{N}-\frac{1}{2}}. \tag{A.48}$$

From (A.47) and (A.48) we have that

$$\|\psi_\epsilon\|_{L^1(\Omega)} \leqslant \|\psi_\epsilon\|_{L^{\frac{2N}{N-2}}(\Omega)}|A_k|^{\frac{N+2}{2N}} \leqslant C\|\psi_\epsilon\|_{H^1(\Omega)}|A_k|^{\frac{N+2}{2N}} \leqslant C\|f_\epsilon\|_{L^p(\Omega)}|A_k|^{\frac{1}{p'}+\frac{2}{N}}.$$

Since, for $p > \frac{N}{2}$ we have that $\frac{1}{p'} + \frac{2}{N} > 1$, it follows from [34, Lemma 5.1] that

$$\|u_\epsilon^2\|_{L^\infty(\Omega)} = \|\tilde{\phi}_\epsilon\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^p(\Omega)} = C\|f_\epsilon\|_{L^p(\Omega)}$$

with $C = C(\Omega, N, p)$.

Step 4. We show that $\|u_\epsilon^1\|_{L^\infty(\Omega)} \leqslant C$ and $\|u_\epsilon^2\|_{L^\infty(R_\epsilon)} \leqslant C$. Observe that by the maximum principle $\|u_\epsilon^1\|_{L^\infty(\Omega)} \leqslant \|u_\epsilon^1\|_{L^\infty(\Gamma_\epsilon^1\cup\Gamma_\epsilon^0)}$ and $\|u_\epsilon^2\|_{L^\infty(R_\epsilon)} \leqslant \|u_\epsilon^2\|_{L^\infty(\Gamma_\epsilon^1\cup\Gamma_\epsilon^0)}$, which both are bounded uniformly in ϵ by the previous steps. □

We are in a position now to provide a complete proof of Proposition 2.7.

Proof of Proposition 2.7. Observe that part (1) follows directly from Lemma A.11.

If $\|f_\epsilon\|_{U_\epsilon^p} \leqslant 1$, then by Lemma A.9 we can get a subsequence, that we denote by ϵ again, and $(f,h) \in U_0^p$, such that if u_ϵ and (w,v) are given by (2.13) and (2.14), then (A.36) holds. In particular, this shows (i). Moreover, from (A.36) we have

$$\|u_\epsilon - w\|_{L^2(\Omega)} + \frac{1}{\epsilon^{(N-1)/2}}\|u_\epsilon - v\|_{L^2(R_\epsilon)} \to 0.$$

Hölder's inequality implies

$$\|u_\epsilon - w\|_{L^1(\Omega)} + \frac{1}{\epsilon^{(N-1)}}\|u_\epsilon - v\|_{L^1(R_\epsilon)} \to 0. \tag{A.49}$$

Since by Lemma A.11 we have $\|u_\epsilon - w\|_{L^\infty(\Omega)} + \|u_\epsilon - v\|_{L^\infty(\Omega)} \leqslant C$, interpolating this estimate with (A.49), we obtain

$$\|u_\epsilon - w\|_{L^q(\Omega)} + \frac{1}{\epsilon^{(N-1)/q}}\|u_\epsilon - v\|_{L^q(R_\epsilon)} \to 0, \quad 1 \leqslant q < \infty.$$

This shows (ii). Finally, (iii) is proved with (i) and (ii) and using a standard cutoff and bootstrap procedure.

The last part of the proposition, statement (3), follows using (2), Lemma A.9 and a standard argument by contradiction. □

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